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8.1 Introduction
Introduction

- A material with **plastic behavior** is characterized by:
  - A nonlinear stress-strain relationship.
  - The existence of permanent (or plastic) strain during a loading/unloading cycle.
  - Lack of unicity in the stress-strain relationship.

- Plasticity is seen in most materials, after an **initial elastic state**.
PRINCIPAL STRESSES

Regardless of the state of stress, it is always possible to choose a special set of axes (principal axes of stress or principal stress directions) so that the shear stress components vanish when the stress components are referred to this system.

The three planes perpendicular to the principle axes are the principal planes.

The normal stress components in the principal planes are the principal stresses.

\[
\begin{bmatrix}
\sigma_1 & 0 & 0 \\
0 & \sigma_2 & 0 \\
0 & 0 & \sigma_3
\end{bmatrix}
\]

\[\sigma_1 \geq \sigma_2 \geq \sigma_3\]
Previous Notions

- **PRINCIPAL STRESSES**
  - The Cauchy stress tensor is a symmetric 2nd order tensor so it will **diagonalize in an orthonormal basis** and its eigenvalues are real numbers.
  - Computing the eigenvalues $\lambda$ and the corresponding eigenvectors $\mathbf{v}$:

  $$\mathbf{\sigma} \cdot \mathbf{v} = \lambda \mathbf{v} \Rightarrow [\mathbf{\sigma} - \lambda \mathbf{1}] \cdot \mathbf{v} = 0$$

  $$\Rightarrow \det [\mathbf{\sigma} - \lambda \mathbf{1}] = |\mathbf{\sigma} - \lambda \mathbf{1}| = \begin{vmatrix} \sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \lambda & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \lambda \end{vmatrix} = 0$$

  - **INVariants**

  $$\lambda^3 - I_1 \lambda^2 - I_2 \lambda - I_3 = 0$$

  $$\begin{cases} 
  \lambda_1 \equiv \sigma_1 \\
  \lambda_2 \equiv \sigma_2 \\
  \lambda_3 \equiv \sigma_3
  \end{cases} \quad \sigma_1 \geq \sigma_2 \geq \sigma_3$$
STRESS INVARIANTS

- Principal stresses are invariants of the stress state.
  - They are invariant w.r.t. rotation of the coordinate axes to which the stresses are referred.
- The principal stresses are combined to form the stress invariants $I$:
  \[
  I_1 = \text{Tr}(\sigma) = \sigma_{ii} = \sigma_1 + \sigma_2 + \sigma_3 \\
  I_2 = \frac{1}{2}(\sigma : \sigma - I_1^2) = -\left(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3\right) \\
  I_3 = \det(\sigma)
  \]

- These invariants are combined, in turn, to obtain the invariants $J$:
  \[
  J_1 = I_1 = \sigma_{ii} \\
  J_2 = \frac{1}{2}(I_1^2 + 2I_2) = \frac{1}{2}\sigma_{ij}\sigma_{ji} = \frac{1}{2}(\sigma : \sigma) \\
  J_3 = \frac{1}{3}(I_1^3 + 3I_1I_2 + 3I_3) = \frac{1}{3}\text{Tr}(\sigma \cdot \sigma \cdot \sigma) = \frac{1}{3}\sigma_{ij}\sigma_{jk}\sigma_{ki}
  \]

REMARK

The $I$ invariants are obtained from the characteristic equation of the eigenvalue problem.

REMARK

The $J$ invariants can be expressed the unified form:
\[
J_i = \frac{1}{i}\text{Tr}\left(\sigma^i\right) \quad i \in \{1, 2, 3\}
\]
SPHERICAL AND DEVIATORIC PARTS OF THE STRESS TENSOR

Given the Cauchy stress tensor $\sigma$ and its principal stresses, the following is defined:

- **Mean stress**
  \[
  \sigma_m = \frac{1}{3} Tr(\sigma) = \frac{1}{3} \sigma_{ii} = \frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3)
  \]

- **Mean pressure**
  \[
  \bar{p} = -\sigma_m = -\frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3)
  \]

- A spherical or **hydrostatic state of stress**: $\sigma_1 = \sigma_2 = \sigma_3 \quad \Rightarrow \quad \sigma = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix} = \sigma 1$

**REMARK**

In a hydrostatic state of stress, the stress tensor is **isotropic** and, thus, its components are the same in any Cartesian coordinate system. As a consequence, any direction is a principal direction and the stress state (traction vector) is the same in any plane.
SPHERICAL AND DEVIATORIC PARTS OF THE STRESS TENSOR

The Cauchy stress tensor $\sigma$ can be split into:

$\sigma = \sigma_{sph} + \sigma'$

- The spherical stress tensor:
  - Also named *mean hydrostatic stress tensor* or *volumetric stress tensor* or *mean normal stress tensor*.
  - Is an isotropic tensor and defines a hydrostatic state of stress.
  - Tends to change the volume of the stressed body.

$$\sigma_{sph} := \sigma_m \mathbf{1} = \frac{1}{3} Tr(\sigma) \mathbf{1} = \frac{1}{3} \sigma_{ii} \mathbf{1}$$

- The stress deviatoric tensor:
  - Is an indicator of how far from a hydrostatic state of stress the state is.
  - Tends to distort the volume of the stressed body.

$$\sigma' = \text{dev} \sigma = \sigma - \sigma_m \mathbf{1}$$
STRESS INVARIANTS OF THE STRESS DEVIATORIC TENSOR

The stress invariants of the stress deviatoric tensor:

\[
I_1' = Tr\left(\sigma'\right) = 0
\]

\[
I_2' = \frac{1}{2}\left(\sigma' : \sigma' - I_1'\right)
\]

\[
I_3' = \det\left(\sigma'\right) = \sigma'_{11}\sigma'_{22}\sigma'_{33} + 2\sigma'_{12}\sigma'_{23}\sigma'_{13} - \sigma'_{12}\sigma'_{23}\sigma'_{13} - \sigma'_{23}\sigma'_{13}\sigma'_{22} = \frac{1}{3}\left(\sigma'_{ij}\sigma'_{jk}\sigma'_{ki}\right)
\]

These correspond exactly with the invariants \( J \) of the same stress deviator tensor:

\[
J_1' = I_1' = 0
\]

\[
J_2' = \frac{1}{2}\left(I_1' + 2I_2'\right) = I_2' = \frac{1}{2}\left(\sigma' : \sigma'\right)
\]

\[
J_3' = \frac{1}{3}\left(I_1' + 3I_2' + 3I_3'\right) = I_3' = \frac{1}{3}\left(I' \cdot \sigma' \cdot \sigma'\right) = \frac{1}{3}\left(\sigma'_{ij}\sigma'_{jk}\sigma'_{ki}\right)
\]
EFFECTIVE STRESS

The effective stress or equivalent uniaxial stress $\bar{\sigma}$ is the scalar:

$$\bar{\sigma} = \sqrt{3J_2} = \sqrt{\frac{3}{2} \sigma'_{ij} \sigma'_{ij}} = \sqrt{\frac{3}{2} \sigma': \sigma'}$$

- It is an invariant value which measures the “intensity” of a 3D stress state in terms of an (equivalent) 1D tensile stress state.
- It should be “consistent”: when applied to a real 1D tensile stress, should return the intensity of this stress.
Example

Calculate the value of the equivalent uniaxial stress for an uniaxial state of stress defined by:

\[
\sigma \equiv \begin{bmatrix}
\sigma_u & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

where \( \sigma_u \) is the uniaxial stress.
Example - Solution

Mean stress: \[ \sigma_m = \frac{1}{3} Tr(\sigma) = \frac{\sigma_u}{3} \]

Spherical and deviatoric parts of the stress tensor:

\[ \sigma_{sph} \equiv \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} = \begin{bmatrix} \frac{\sigma_u}{3} & 0 & 0 \\ 0 & \frac{\sigma_u}{3} & 0 \\ 0 & 0 & \frac{\sigma_u}{3} \end{bmatrix} \]

\[ \sigma' = \sigma - \sigma_{sph} \equiv \begin{bmatrix} \sigma_u - \sigma_m & 0 & 0 \\ 0 & -\sigma_m & 0 \\ 0 & 0 & -\sigma_m \end{bmatrix} = \begin{bmatrix} 2 \sigma_u & 0 & 0 \\ 0 & -\frac{1}{3} \sigma_u & 0 \\ 0 & 0 & -\frac{1}{3} \sigma_u \end{bmatrix} \]

\[ \bar{\sigma} = \sqrt{\frac{3}{2} \sigma_{ij}' \sigma_{ij}'} = \sqrt{\frac{3}{2} \sigma_u^2 \left( \frac{4}{9} + \frac{1}{9} + \frac{1}{9} \right)} = \sqrt{\frac{3}{2} \frac{2}{3}} |\sigma_u| = \sqrt{\frac{3}{2} |\sigma_u|} \]

\[ \bar{\sigma} = |\sigma_u| \]
8.2 Principal Stress Space
The principal stress space or Haigh–Westergaard stress space is the space defined by a system of Cartesian axes where the three spatial axes represent the three principal stresses for a body subject to stress:

\[ \sigma_1 \geq \sigma_2 \geq \sigma_3 \]
Any of the planes perpendicular to the hydrostatic stress axis is an octahedral plane.

- Its unit normal is \( \mathbf{n} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \).

\[ \sigma_1 \geq \sigma_2 \geq \sigma_3 \]
Consider the principal stress space:

- The normal octahedral stress is defined as:

\[
\sqrt{3} \sigma_{oct} = |OA| = \overline{OP} \cdot \mathbf{n} = \left[ \sigma_1, \sigma_2, \sigma_3 \right] \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \\
\frac{\sqrt{3}}{3} \left( \sigma_1 + \sigma_2 + \sigma_3 \right) = \sqrt{3} \sigma_m
\]

\[
\sigma_{oct} = \sigma_m = \frac{I_1}{3}
\]
Consider the principal stress space:

- The shear or tangential octahedral stress is defined as:
  \[ \sqrt{3} \tau_{oct} = \left| AP \right| \]

Where the \( \left| AP \right| \) is calculated from:

\[
3 \tau_{oct}^2 = \left| AP \right|^2 = OP^2 - OA^2 = \left( \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \right) - \left( \frac{1}{3} \left( \sigma_1 + \sigma_2 + \sigma_3 \right)^2 \right) = 2 J_2^\prime
\]

Alternative forms of \( \tau_{oct} \):

\[
\tau_{oct} = \frac{1}{\sqrt{3}} \left[ \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \frac{1}{3} \left( \sigma_1 + \sigma_2 + \sigma_3 \right)^2 \right]^{1/2}
\]

\[
\tau_{oct} = \frac{1}{3\sqrt{3}} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 \right]^{1/2}
\]
In a pure spherical stress state:

\[ \sigma = \sigma 1 \rightarrow \sigma_m = \frac{1}{3} 3\sigma \rightarrow \sigma_{esf} = \sigma 1 = \sigma \]

\[ \sigma' = \sigma - \sigma_{esf} = 0 \]

\[ J_2' = 0 \]

\[ \tau_{oct} = 0 \]

A pure spherical stress state is located on the hydrostatic stress axis.

In a pure deviator stress state:

\[ \sigma = \sigma' \]

\[ \sigma_m = Tr(\sigma) = Tr(\sigma') = 0 \]

\[ \sigma_{oct} = 0 \]

A pure deviator stress state is located on the octahedral plane containing the origin of the principal stress space.
Any point in space is unambiguously defined by the three invariants:

- **The first stress invariant** \( I_1 \) characterizes the distance from the origin to the octahedral plane containing the point.

- **The second deviator stress invariant** \( J'_2 \) characterizes the radius of the cylinder containing the point and with the hydrostatic stress axis as axis.

- **The third deviator stress invariant** \( J'_3 \) characterizes the position of the point on the circle obtained from the intersection of the octahedral plane and the cylinder. It defines an angle \( \theta(J'_3) \).
The projection of the principal stress space on the octahedral plane results in the division of the plane into six “sectors”:

- These are characterized by the different principal stress orders.

Election of a criterion, e.g.: $\sigma_1 \geq \sigma_2 \geq \sigma_3$
8.4 Phenomenological Behaviour

Ch.8. Plasticity
Notion of Plastic Strain

\[ \varepsilon = \varepsilon^e + \varepsilon^p \]

- Elastic limit: \( \sigma_e \)
- Linear elastic behaviour: \( \sigma = E \varepsilon^e \)

PLASTIC STRAIN
Bauschinger Effect

- Also known as **kinematic hardening**.
Elastoplastic Behaviour

Considering the phenomenological behaviour observed, elastoplastic materials are characterized by:

- Lack of unicity in the stress-strain relationship.
  - The stress value depends on the actual strain and the previous loading history.

- A nonlinear stress-strain relationship.
  - There may be certain phases in the deformation process with incremental linearity.

- The existence of permanent (or plastic) strain during a loading / unloading cycle.
8.5 1D Incremental Plasticity Theory

Ch.8. Plasticity
Introduction

- The **incremental plasticity theory** is a mathematical model used to represent the evolution of the stress-strain curve in an elastoplastic material.
  - Developed for 1D but it can be generalized for 3D problems.

**Remark**
This theory is developed under the hypothesis of infinitesimal strains.
Additive Decomposition of Strain

- Total strain can be split into an **elastic** (recoverable) part, $\varepsilon^e$, and an **inelastic** (unrecoverable) one, $\varepsilon^p$:

  \[
  \varepsilon = \varepsilon^e + \varepsilon^p
  \]

  where \( \varepsilon^e = \frac{\sigma}{E} \) elastic modulus or Young modulus

- Also,

  \[
  d\varepsilon = d\varepsilon^e + d\varepsilon^p \quad \text{where} \quad d\varepsilon^e = \frac{d\sigma}{E}
  \]
Hardening Variable

- The **hardening variable**, \( \alpha \), is defined as:

\[
d\alpha = \text{sign}(\sigma) d\varepsilon^p
\]

Such that \( d\alpha \geq 0 \) and \( \alpha|_{\varepsilon^p=0} = 0 \).

- Note that \( \alpha \) is always positive and:

\[
d\alpha = \left| d\alpha \right| = \left| \text{sign}(\sigma) \right| d\varepsilon^p = 1
\]

Then, for a monotonously increasing plastic strain process, both variables coincide:

\[
d\varepsilon^p \geq 0 \quad \Rightarrow \quad \alpha = \int_0^{\varepsilon^p} |d\varepsilon^p| = \int_0^{\varepsilon^p} d\varepsilon^p = \varepsilon^p
\]
Stress value, $\sigma_f$, threshold for the material exhibiting plastic behaviour after elastic unloading + elastic loading

- It is considered a material property.
- For $\epsilon^p = \alpha = 0 \quad \Rightarrow \quad \sigma_f = \sigma_e$

Yield Stress and Hardening Law

$\sigma = \sigma_f(\alpha)$

$\sigma_f = \sigma_e + H'\alpha$

$H'$ is the hardening modulus
The yield function, \( F(\sigma, \alpha) \), characterizes the state of the material:

\[
F(\sigma, \alpha) \equiv |\sigma| - \sigma_f(\alpha)
\]

- \( F(\sigma, \alpha) < 0 \): ELASTIC STATE
- \( F(\sigma, \alpha) = 0 \): ELASTO-PLASTIC STATE

Space of admissible stresses

\[
\mathbb{E}_\sigma := \{ \sigma \in \mathbb{R} \mid F(\sigma, \alpha) < 0 \}
\]

ELASTIC DOMAIN

\[
\partial \mathbb{E}_\sigma := \{ \sigma \in \mathbb{R} \mid F(\sigma, \alpha) = 0 \}
\]

YIELD SURFACE

INITIAL ELASTIC DOMAIN:

\[
\mathbb{E}_\sigma^0 := \{ \sigma \in \mathbb{R} \mid F(\sigma, 0) \equiv |\sigma| - \sigma_e < 0 \}
\]
Any admissible stress state must belong to the space of admissible stresses, $\overline{E}_\sigma$ (postulate):

$$\overline{E}_\sigma = E_\sigma \cup \partial E_\sigma = \{ \sigma \in \mathbb{R} \mid F(\sigma, \alpha) \leq 0 \}$$

$$F(\sigma, \alpha) \equiv |\sigma| - \sigma_f(\alpha)$$

**REMARK**

$$\overline{E}_\sigma \equiv [-\sigma_f(\alpha), \sigma_f(\alpha)]$$
The following situations are defined:

- **ELASTIC REGIME**
  \[ \sigma \in \mathbb{E}_\sigma \quad \Rightarrow \quad d\sigma = E \, d\varepsilon \]

- **ELASTOPLASTIC REGIME – UNLOADING**
  \[ \sigma \in \partial \mathbb{E}_\sigma \quad \text{and} \quad dF(\sigma, \alpha) < 0 \]
  \[ d\sigma = E \, d\varepsilon \]

- **ELASTOPLASTIC REGIME – PLASTIC LOADING**
  \[ \sigma \in \partial \mathbb{E}_\sigma \quad \text{and} \quad dF(\sigma, \alpha) = 0 \]
  \[ d\sigma = E^{ep} \, d\varepsilon \]

**REMARK**

The situation \[ \sigma \in \partial \mathbb{E}_\sigma \quad \text{and} \quad dF(\sigma, \alpha) > 0 \]

is not possible because, by definition, on the yield surface \( F(\sigma, \alpha) = 0 \).
Consider the elastoplastic regime in plastic loading,

\[ \sigma \in \partial E_{\sigma} \]

\[ F(\sigma, \alpha) \equiv |\sigma| - \sigma_f(\alpha) = 0 \quad \Rightarrow \quad dF(\sigma, \alpha) = 0 \]

\[ dF(\sigma, \alpha) = \frac{\partial |\sigma|}{\partial \sigma} d\sigma - \sigma'_f(\alpha) d\alpha = 0 \quad \Rightarrow \]

\[ d\alpha = \frac{1}{H'} \text{sign}(\sigma) d\sigma \]

Since the hardening variable is defined as:

\[ d\alpha = \text{sign}(\sigma) d\varepsilon^p \]

\[ d\varepsilon^p = \frac{1}{H'} d\sigma \quad \text{for} \quad \sigma \in \partial E_{\sigma} \]
Elastoplastic Tangent Modulus

Elastic strain $\Rightarrow d\varepsilon^e = \frac{1}{E} \ d\sigma$

Plastic strain $\Rightarrow d\varepsilon^p = \frac{1}{H'} \ d\sigma$

Additive strain decomposition:

$$d\varepsilon = d\varepsilon^e + d\varepsilon^p = \left(\frac{1}{E} + \frac{1}{H'}\right)d\sigma$$

$$d\sigma = \frac{1}{E + H'} \ d\varepsilon = \frac{EH'}{E + H'} \ d\varepsilon$$

$$d\sigma = E^{ep} \ d\varepsilon$$

ELASTOPLASTIC TANGENT MODULUS

$$E^{ep} = \frac{EH'}{E + H'}$$
Following the **constitutive equation** defined

**Elastoplastic Regime**  
\[ d\sigma = E^{ep} \, d\varepsilon \]

**Elastic Regime**  
\[ d\sigma = E \, d\varepsilon \]

**Remark**  
Plastic strain is generated only during the plastic loading process.
The value of the hardening modulus, $H'$, determines the following situations:

- **Linear elasticity** ($H' > 0$)
- **Plasticity with strain hardening** ($H' > 0$)
- **Perfect plasticity** ($H' = 0$)
- **Plasticity with strain softening** ($H' < 0$)

The modulus of the hardening, $H'$, is defined as:

$$E^{ep} = \frac{EH'}{E + H'}$$
In real materials, the stress-strain curve shows a combination of the three types of hardening modulus.
8.6 3D Incremental Theory

Ch.8. Plasticity
The 1D incremental plasticity theory can be generalized to a multiaxial stress state in 3D.

The same concepts are used:
- Additive decomposition of strain
- Hardening variable
- Yield function

Plus, additional ones are added:
- Loading - unloading conditions
- Consistency conditions
Total strain can be split into an **elastic** (recoverable) part, $\varepsilon^e$, and an **inelastic** (unrecoverable) one, $\varepsilon^p$:

$$\varepsilon = \varepsilon^e + \varepsilon^p$$

where $\varepsilon^e = \mathbf{C}^{-1} : \sigma$

Also,

$$d\varepsilon = d\varepsilon^e + d\varepsilon^p$$

where $d\varepsilon^e = \mathbf{C}^{-1} : d\sigma$
The hardening variable, \( \alpha = f(\sigma, \varepsilon^p) \), is a scalar:

\[
d\alpha = \lambda \quad \text{with} \quad \alpha \in [0, \infty)
\]

Where \( \lambda \) is known as the plastic multiplier.

The flow rule is defined as:

\[
d\varepsilon^p = \lambda \frac{\partial G(\sigma, \alpha)}{\partial \sigma}
\]

Where \( G(\sigma, \alpha) \) is the plastic potential function.
The yield function, $F(\sigma, \alpha)$, is a scalar defined as:

$$F(\sigma, \alpha) = \phi(\sigma) - \sigma_f(\alpha)$$

1D $\to F(\sigma, \alpha) \equiv |\sigma| - \sigma_f(\alpha)$

- ELASTIC STATE
  - $F(\sigma, \alpha) < 0$
  - $F(\sigma, \alpha) \in \mathbb{E}_\sigma := \{\sigma \mid F(\sigma, \alpha) < 0\}$
  - $\mathbb{E}_\sigma$ is the ELASTIC DOMAIN

- ELASTOPLASTIC STATE
  - $F(\sigma, \alpha) = 0$
  - $\partial\mathbb{E}_\sigma := \{\sigma \mid F(\sigma, \alpha) = 0\}$
  - $\partial\mathbb{E}_\sigma$ is the YIELD SURFACE

INITIAL ELASTIC DOMAIN:

$\mathbb{E}_\sigma^0 := \{\sigma \mid F(\sigma, 0) < 0\}$

Space of admissible stresses:

$$\overline{\mathbb{E}}_\sigma = \mathbb{E}_\sigma \cup \partial\mathbb{E}_\sigma$$
Loading-Unloading Conditions and Consistency Condition

- **Loading/unloading conditions** (also known as Karush-Kuhn-Tucker conditions):
  \[
  \lambda \geq 0 \quad ; \quad F(\sigma, \alpha) \leq 0 \quad ; \quad \lambda F(\sigma, \alpha) = 0
  \]

- **Consistency conditions**:
  For \( F(\sigma, \alpha) = 0 \) → \( \lambda dF(\sigma, \alpha) = 0 \)

- \( F = 0; \ dF < 0 \quad \Rightarrow \quad \lambda = 0; \quad d\varepsilon^p = \lambda \frac{\partial G(\sigma, \alpha)}{\partial \sigma} = 0 \quad \Rightarrow \quad \text{ELASTOPLASTIC ELASTIC UNLOADING} \)

- \( F = 0; \ dF = 0 \quad \Rightarrow \quad \begin{cases} \lambda = 0; & d\varepsilon^p = \lambda \frac{\partial G(\sigma, \alpha)}{\partial \sigma} = 0 \\ \lambda > 0; & d\varepsilon^p = \lambda \frac{\partial G(\sigma, \alpha)}{\partial \sigma} \neq 0 \end{cases} \quad \Rightarrow \quad \text{ELASTOPLASTIC NEUTRAL LOADING} \)

- \( F = 0; \ dF > 0 \quad \Rightarrow \quad \text{IMPOSSIBLE} \)

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Constitutive Equation

- The following situations are defined:
  - **ELASTIC REGIME** \((F < 0)\)
    \[
    \sigma \in E_\sigma \quad \Rightarrow \quad d\sigma = C : d\varepsilon
    \]
  - **ELASTOPLASTIC REGIME – ELASTIC UNLOADING** \((F = 0\) \text{ and } dF(\sigma, \alpha) < 0)\)
    \[
    \sigma \in \partial E_\sigma \quad \text{and} \quad dF(\sigma, \alpha) < 0
    \]
    \[
    d\sigma = C : d\varepsilon
    \]
  - **ELASTOPLASTIC REGIME – PLASTIC LOADING** \((F = 0\) \text{ and } dF(\sigma, \alpha) = 0)\)
    \[
    \sigma \in \partial E_\sigma \quad \text{and} \quad dF(\sigma, \alpha) = 0
    \]
    \[
    d\sigma = C^{ep} : d\varepsilon
    \]

ELASTOPLASTIC CONSTITUTIVE TENSOR
The elastoplastic constitutive tensor is written as:

\[
\mathbf{C}^{ep} (\sigma, \alpha) = \mathbf{C} - \mathbf{C} : \frac{\partial G}{\partial \sigma} \otimes \frac{\partial F}{\partial \sigma} : \mathbf{C} \\
H' + \frac{\partial F}{\partial \sigma} : \mathbf{C} : \frac{\partial G}{\partial \sigma}
\]

\[
\mathbf{C}^{ep}_{ijkl} = \mathbf{C}_{ijkl} - \mathbf{C}_{ijpq} \frac{\partial G}{\partial \sigma_{pq}} \frac{\partial F}{\partial \sigma_{rs}} \mathbf{C}_{rskl} \\
H' + \frac{\partial F}{\partial \sigma_{pq}} \mathbf{C}_{pqrs} \frac{\partial G}{\partial \sigma_{rs}}
\]

\[i, j, k, l, p, q, r, s \in \{1, 2, 3\}\]

**REMARK**

When the plastic potential function and the yield function coincide, it is said that there is **associated flow**:

\[
G(\sigma, \alpha) = F(\sigma, \alpha)
\]
8.7 Failure Criteria: Yield Surfaces

Ch.8. Plasticity
The initial **yield surface**, \( \partial \mathcal{E}_0^0 \), is the external boundary of the initial elastic domain \( \mathcal{E}_0^0 \) for the virgin material.

- The state of stress **inside** the yield surface is elastic for the virgin material.
- When in a deformation process, the stress state reaches the yield surface, the virgin material losses elasticity for the first time: this is considered as a **failure criterion** for design. Subsequent stages in the deformation process are not considered.

\[
\partial \mathcal{E}_0^0 := \{\sigma \mid \phi(\sigma) = \sigma_e\}
\]

\[
\mathcal{E}_0^0 := \{\sigma \mid \phi(\sigma) < \sigma_e\}
\]
The yield surface is usually expressed in terms of the following invariants to make it independent of the reference system (in the principal stress space):

\[ F(\sigma) \equiv F(I_1, J'_2, J'_3) - \sigma_e = 0 \]

where:

\[ I_1 = Tr(\sigma) = \sigma_{ii} = \sigma_1 + \sigma_2 + \sigma_3 \]
\[ J'_2 = \frac{1}{2}(I'_1 + 2I'_2) = I'_2 = \frac{1}{2}(\sigma' : \sigma') \]
\[ J'_3 = \frac{1}{3}(I'_1 + 3I'_2) = I'_3 = \frac{1}{3} Tr(\sigma' \cdot \sigma' \cdot \sigma') = \frac{1}{3} (\sigma'_{ij} \sigma'_{jk} \sigma'_{ki}) \]

The elastoplastic behavior will be isotropic.

\[ \phi(\sigma) \]

with \( \sigma_1 \geq \sigma_2 \geq \sigma_3 \)

**REMARK**

Due to the adopted principal stress criteria, the definition of yield surface only affects the first sector of the principal stress space.
Von Mises Criterion

- The yield surface is defined as:

\[ F(\sigma) \equiv \bar{\sigma}(\sigma) - \sigma_e = 0 \]

Where \( \bar{\sigma}(\sigma) = \sqrt{3J'_2} \) is the effective stress. (often termed the Von-Mises stress)

- The shear octahedral stress is, by definition, \( \tau_{oct} = \sqrt{\frac{2}{3} [J'_2]^{1/2}} \).

Thus, the effective stress is rewritten:

\[ [J'_2]^{1/2} = \sqrt{\frac{3}{2}} \tau_{oct} \quad \Rightarrow \quad \bar{\sigma}(\sigma) = \sqrt{3} \sqrt{\frac{3}{2}} \tau_{oct} = \frac{3}{\sqrt{2}} \tau_{oct} \]

- And the yield surface is given by:

\[ F(\sigma) \equiv \frac{3}{\sqrt{2}} \tau_{oct} - \sigma_e = 0 \]

REMARK
The Von Mises criterion depends solely on the second deviator stress invariant.
The octahedral stresses characterize the radius of the cylinder containing the point and with the hydrostatic stress axis as axis.

\[ F(\sigma) = \frac{3}{\sqrt{2}} \tau_{oct} - \sigma_e = 0 \]

**Remark**

The Von Mises Criterion is adequate for metals, where hydrostatic stress states have an elastic behavior and failure is typically due to deviatoric stress components.
Example

Consider a beam under a composed flexure state such that for a beam section the stress state takes the form,

\[
\sigma = \begin{bmatrix}
\sigma_x & \tau_{xy} & 0 \\
\tau_{xy} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Obtain the expression for Von Mises criterion.
Example - Solution

The mean stress is:

\[ \sigma_m = \frac{1}{3} Tr(\sigma) = \frac{\sigma_x}{3} \]

The deviator part of the stress tensor is:

\[ \sigma' = \sigma - \sigma_{esf} \equiv \begin{bmatrix} \sigma_x - \sigma_m & \tau_{xy} & 0 \\ \tau_{xy} & -\sigma_m & 0 \\ 0 & 0 & -\sigma_m \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & -\frac{1}{3} \sigma_x & 0 \\ 0 & 0 & -\frac{1}{3} \sigma_x \end{bmatrix} \]

The second deviator stress invariant is given by,

\[ J'_2 = \frac{1}{2} \sigma' : \sigma' = \frac{1}{2} \left( \frac{4}{9} \sigma_x^2 + \frac{1}{9} \sigma_x^2 + \frac{1}{9} \sigma_x^2 + \tau_{xy}^2 + \tau_{xy}^2 \right) = \frac{1}{3} \sigma_x^2 + \tau_{xy}^2 \]
Example - Solution

The uniaxial effective stress is:

\[ \bar{\sigma}(\sigma) = \sqrt{3J'_2} = \sqrt{\sigma_x^2 + 3\tau_{xy}^2} \]

Finally, the Von Mises yield surface is given by the expression:

\[ F(\sigma) \equiv \sqrt{3J'_2} - \sigma_e = 0 \]

\[ \sqrt{\sigma_x^2 + 3\tau_{xy}^2} = \sigma_e \]

(Criterion in design codes for metal beams)
Also known as the **maximum shear stress criterion**, it establishes that the elastic domain ends when:

\[ \tau_{\text{max}} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_e}{2} \]

It can be written univocally in terms of invariants \( J'_2 \) and \( J'_3 \):

\[ F(\sigma) \equiv (\sigma_1 - \sigma_3) - \sigma_e \equiv F(J'_2, J'_3) - \sigma_e = 0 \]
The Tresca yield surface is appropriate for metals, which have an elastic behavior under hydrostatic stress states and basically have the same traction/compression behavior.
Example

Obtain the expression of the Tresca criterion for an uniaxial state of stress defined by:

\[
\sigma = \begin{bmatrix}
\sigma_u & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]
Consider:

\[ \sigma_u \geq 0 \quad \Rightarrow \quad \begin{cases} 
\sigma_1 = \sigma_u \\
\sigma_3 = 0 
\end{cases} \quad \Rightarrow \quad F(\sigma) = (\sigma_1 - \sigma_3) - \sigma_e = \sigma_u - \sigma_e = \sqrt{\sigma_u^2 - \sigma_e^2} \\
\]

\[ \sigma_u < 0 \quad \Rightarrow \quad \begin{cases} 
\sigma_1 = 0 \\
\sigma_3 = \sigma_u 
\end{cases} \quad \Rightarrow \quad F(\sigma) = (\sigma_1 - \sigma_3) - \sigma_e = -\sigma_u - \sigma_e = \sqrt{\sigma_u^2 - \sigma_e^2} \\
\]

The Tresca criterion is expressed as:

\[ F(\sigma) \equiv \bar{\sigma} - \sigma_e = 0 \quad \Rightarrow \quad \left| \sigma_u \right| = \sigma_e \]

Note that it coincides with the Von Mises criterion for an uniaxial state of stress.
Example

Consider a beam under a composed flexure state such that for a beam section the stress state takes the form,

\[
[\sigma] = \begin{bmatrix}
\sigma_x & \tau_{xy} & 0 \\
\tau_{xy} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Obtain the expression for Tresca yield surface.
Example - Solution

The principal stresses are:

\[ \sigma_1 = \frac{1}{2} \sigma_x + \sqrt{\frac{1}{4} \sigma_x^2 + \tau_{xy}^2} \], \quad \sigma_3 = \frac{1}{2} \sigma_x - \sqrt{\frac{1}{4} \sigma_x^2 + \tau_{xy}^2} \]

Taking the definition of the Tresca yield surface,

\[ F(\sigma) \equiv (\sigma_1 - \sigma_3) - \sigma_e = 0 \]

\[ \sigma_e = \sigma_1 - \sigma_3 = \left( \frac{1}{2} \sigma_x + \sqrt{\frac{1}{4} \sigma_x^2 + \tau_{xy}^2} \right) - \left( \frac{1}{2} \sigma_x - \sqrt{\frac{1}{4} \sigma_x^2 + \tau_{xy}^2} \right) \]

\[ \sqrt{\sigma_x^2 + 4 \tau_{xy}^2} = \sigma_e \]

(\text{comparison stress})

\[ \sigma_{co} \]
**Mohr-Coulomb Criterion**

- It is a generalization of the Tresca criterion, by including the influence of the first stress invariant.
- In the Mohr circle’s plane, the Mohr-Coulomb yield function takes the form,

\[ \tau = c - \sigma \tan \phi \]

**REMARK**

The yield line cuts the normal stress axis at a positive value, limiting the materials tensile strength.
Möhr-Coulomb Criterion

Consider the stress state for which the yield point is reached:

\[ \tau_A = R \cos \phi \]
\[ \sigma_A = \frac{\sigma_1 + \sigma_3}{2} + R \sin \phi \]

\[ \tau_A + \sigma_A \tan \phi - c = 0 \]
\[ (\sigma_1 - \sigma_3) + (\sigma_1 + \sigma_3) \sin \phi - 2c \cos \phi = 0 \]

\[ F(\sigma) \equiv (\sigma_1 - \sigma_3) + (\sigma_1 + \sigma_3) \sin \phi - 2c \cos \phi = 0 \]

**Remark**
For \( \phi = 0 \) and \( c = \sigma_e / 2 \), the Tresca criterion is recovered.
Mohr-Coulomb Criterion

*Remark*

The Mohr-Coulomb yield surface is appropriate for frictional cohesive materials, such as concrete, soils or rocks which have considerably different tensile and compressive values for the uniaxial elastic limit.
Drucker-Prager Criterion

- It is a generalization of the Von Mises criterion, by including the influence of the first stress invariant.

- The yield surface is given by the expression:

\[ F(\sigma) \equiv 3\alpha\sigma_m + [J'_2]^{1/2} - \beta = 0 \]

Where:
\[ \alpha = \frac{2\sin \phi}{\sqrt{3} (3 - \sin \phi)} \quad ; \quad \beta = \frac{6c \cos \phi}{\sqrt{3} (3 - \sin \phi)} \quad ; \quad \sigma_m = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \frac{I_1}{3} \]

- It can be rewritten as:

\[ F(\sigma) \equiv \alpha I_1 + [J'_2]^{1/2} - \beta = 3\alpha\sigma_{oct} + \sqrt{\frac{3}{2}} \tau_{oct} - \beta = F(I'_1, J'_2) \]

**REMARK**

For \( \phi = 0 \) and \( c = \sigma_e / 2 \), the Von Mises criterion is recovered.
The Drucker-Prager yield surface, like the Mohr-Coulomb one, is appropriate for frictional cohesive materials, such as concrete, soils or rocks which have considerably different tensile and compressive values for the uniaxial elastic limit.
Chapter 8
Plasticity

8.1 Introduction
The elastoplastic models (constitutive equations) are used in continuum mechanics to represent the mechanical behavior of materials whose behavior, once certain limits in the values of the stresses (or strains) are exceeded, is no longer representable by means of simpler models such as the elastic ones. In this chapter, these models will be studied considering, in all cases, that strains are infinitesimal.

Broadly speaking, plasticity introduces two important modifications with respect to the linear elasticity seen in chapters 6 and 7:

1) The loss of linearity: stresses cease to be proportional to strains.
2) The concept of permanent or plastic strain: a portion of the strain generated during the loading process is not recovered during the unloading process.

8.2 Previous Notions
The concepts in this section are a review of those already studied in Sections 4.4.4 to 4.4.7 of Chapter 4.

8.2.1 Stress Invariants
Consider the Cauchy stress tensor $\sigma$ and its matrix of components in a base associated with the Cartesian axes $\{x, y, z\}$ (see Figure 8.1),

$$\sigma_{\text{xyz}} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}.$$  \hspace{1cm} (8.1)
Since $\mathbf{\sigma}$ is a symmetrical second-order tensor, it will diagonalize in an orthonormal base and all its eigenvalues will be real numbers. Then, consider a system of Cartesian axes $\{x', y', z'\}$ associated with a base in which $\mathbf{\sigma}$ diagonalizes. Its matrix of components in this base is

$$
[\mathbf{\sigma}]_{x'y'z'} = 
\begin{bmatrix}
\sigma_1 & 0 & 0 \\
0 & \sigma_2 & 0 \\
0 & 0 & \sigma_3
\end{bmatrix}
$$

(8.2)

where $\sigma_1 \geq \sigma_2 \geq \sigma_3$, denoted as principal stresses, are the eigenvectors of $\mathbf{\sigma}$ and the directions associated with the axes $\{x', y', z'\}$ are named principal directions (see Figure 8.1).

To obtain the stresses and the principal directions of $\mathbf{\sigma}$, the corresponding eigenvalue and eigenvector problem must be solved:

Find $\lambda$ and $\mathbf{v}$ such that $\mathbf{\sigma} \cdot \mathbf{v} = \lambda \mathbf{v} \Rightarrow (\mathbf{\sigma} - \lambda \mathbf{1}) \cdot \mathbf{v} = 0$,  

(8.3)

where $\lambda$ corresponds to the eigenvalues and $\mathbf{v}$ to the eigenvectors. The necessary and sufficient condition for (8.3) to have a solution is

$$
\det(\mathbf{\sigma} - \lambda \mathbf{1}) = |\mathbf{\sigma} - \lambda \mathbf{1}| = 0 ,
$$

(8.4)

which, in component form, results in

$$
\begin{vmatrix}
\sigma_x - \lambda & \tau_{xy} & \tau_{xz} \\
\tau_{xy} & \sigma_y - \lambda & \tau_{yz} \\
\tau_{xz} & \tau_{yz} & \sigma_z - \lambda
\end{vmatrix} = 0 .
$$

(8.5)
The algebraic development of (8.5), named characteristic equation, corresponds to a third-degree polynomial equation in \( \lambda \), that can be written as

\[
\lambda^3 - I_1 \lambda^2 - I_2 \lambda - I_3 = 0, \tag{8.6}
\]

where the coefficients \( I_1 (\sigma_{ij}) \), \( I_2 (\sigma_{ij}) \) and \( I_3 (\sigma_{ij}) \) are certain functions of the components \( \sigma_{ij} \) of the tensor \( \sigma \) expressed in the coordinate system \( \{x, y, z\} \). Yet, the solutions to (8.6), which will be a function of its coefficients \( (I_1, I_2, I_3) \), are the principal stresses that, on the other hand, are independent of the system of axes chosen to express \( \sigma \). Consequently, said coefficients must be invariant with respect to any change of base. Therefore, the coefficients \( I_1, I_2 \) and \( I_3 \) are denoted as \( I \) stress invariants or fundamental stress invariants and their expression (resulting from the computation of (8.5)) is

\[
\begin{align*}
I_1 &= \text{Tr}(\sigma) = \sigma_{ii} = \sigma_1 + \sigma_2 + \sigma_3, \\
I_2 &= \frac{1}{2} (\sigma : \sigma - I_1^2) = -\left(\sigma_1 \sigma_2 + \sigma_1 \sigma_3 + \sigma_2 \sigma_3\right), \\
I_3 &= \det(\sigma) = \sigma_1 \sigma_2 \sigma_3.
\end{align*}
\tag{8.7}
\]

Obviously, any scalar function of the stress invariants will also be an invariant and, thus, new invariants can be defined based on the \( I \) stress invariants given in (8.7). In particular, the so-called \( J \) stress invariants are defined as

\[
\begin{align*}
J_1 &= I_1 = \sigma_{ii} = \text{Tr}(\sigma), \\
J_2 &= \frac{1}{2} \left(I_1^2 + 2I_2\right) = \frac{1}{2} \sigma_{ij} \sigma_{ji}, \\
J_3 &= \frac{1}{3} \left(I_1^3 + 3I_1I_2 + 3I_3\right) = \frac{1}{3} \sigma_{ij} \sigma_{jk} \sigma_{ki},
\end{align*}
\tag{8.8}
\]

where \( \{x, y, z\} \) is the coordinate system and \( \sigma \) is the stress tensor expressed in this system.
Remark 8.1. Note that
\[ I_1 = 0 \implies J_i = I_i \quad i \in \{1, 2, 3\} \, . \]
Also, the invariants \( J_i \), \( i \in \{1, 2, 3\} \) can be expressed in a unified and compact form by means of
\[ J_i = \frac{1}{i} \text{Tr} (\sigma^i) \quad i \in \{1, 2, 3\} \, . \]

8.2.2 Spherical and Deviatoric Components of the Stress Tensor

Given the stress tensor \( \sigma \), the mean stress \( \sigma_m \) is defined as
\[ \sigma_m = \frac{I_1}{3} = \frac{1}{3} \text{Tr} (\sigma) = \frac{1}{3} \sigma_{ii} = \frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3) \, . \tag{8.9} \]
and the mean pressure \( \bar{p} \) as
\[ \bar{p} = -\sigma_m \, . \tag{8.10} \]

The Cauchy stress tensor can be decomposed into a spherical part (or component), \( \sigma_{\text{sph}} \), and a deviatoric one, \( \sigma' \),
\[ \sigma = \sigma_{\text{sph}} + \sigma' \, . \tag{8.11} \]
where the spherical part of the stress tensor is defined as
\[ \sigma_{\text{sph}} \overset{\text{def}}{=} \frac{1}{3} \text{Tr} (\sigma) 1 = \sigma_m 1 \]
\[ \sigma_{\text{sph not}} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} \, . \tag{8.12} \]
and, from (8.11) and (8.12), the deviatoric part is given by
\[ \sigma' = \sigma - \sigma_{\text{sph not}} = \begin{bmatrix} \sigma_x - \sigma_m & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma_m & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma_m \end{bmatrix} \, . \tag{8.13} \]
Finally, the $I$ and $J$ invariants of the deviatoric tensor $\sigma'$, named $I'$ and $J'$ invariants, respectively, are derived from (8.7), (8.8), (8.9) and (8.13).

\[
\begin{align*}
J' \text{ stress invariants} & \quad \begin{cases} 
J'_1 = I'_1 = 0 \\
J'_2 = I'_2 = \frac{1}{2} (\sigma' : \sigma') = \frac{1}{2} \sigma'_{ij} \sigma'_{ji} \\
J'_3 = I'_3 = \frac{1}{3} (\sigma'_{ij} \sigma'_{jk} \sigma'_{ki}) 
\end{cases} 
\end{align*}
\] (8.14)

Remark 8.2. It is easily proven that the principal directions of $\sigma$ coincide with those of $\sigma'$, that is, that both tensors diagonalize in the same base. In effect, working in the base associated with the principal directions of $\sigma$, i.e., the base in which $\sigma$ diagonalizes, and, given that $\sigma_{sph}$ is a hydrostatic tensor and, thus, is diagonal in any base, then $\sigma'$ also diagonalizes in the same base (see Figure 8.2).

Figure 8.2: Diagonalization of the spherical and deviatoric parts of the stress tensor.
Remark 8.3. The effective stress or equivalent uniaxial stress $\bar{\sigma}$ is the scalar

$$\bar{\sigma} = \sqrt{3J_2} = \frac{\sqrt{3}}{2} \sigma'_{ij} \sigma'_{ji} = \frac{\sqrt{3}}{2} \sigma' : \sigma'. $$

The name of equivalent uniaxial stress is justified because its value for an uniaxial stress state coincides with said uniaxial stress (see Example 8.1).

**Example 8.1** – Compute the value of the equivalent uniaxial stress (or effective stress) $\bar{\sigma}$ for an uniaxial stress state defined by

$$\sigma_{\text{not}} = \begin{bmatrix} \sigma_u & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. $$

**Solution**

The mean stress is

$$\sigma_m = \frac{1}{3} \text{Tr}(\sigma) = \frac{\sigma_u}{3}. $$

Then, the spherical component of the stress tensor is

$$\sigma_{\text{sph}}_{\text{not}} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} = \begin{bmatrix} \frac{\sigma_u}{3} & 0 & 0 \\ 0 & \frac{\sigma_u}{3} & 0 \\ 0 & 0 & \frac{\sigma_u}{3} \end{bmatrix}. $$

and the deviatoric component results in

$$\sigma' = \sigma - \sigma_{\text{sph}}_{\text{not}} = \begin{bmatrix} \sigma_u - \sigma_m & 0 & 0 \\ 0 & -\sigma_m & 0 \\ 0 & 0 & -\sigma_m \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \sigma_u & 0 & 0 \\ 0 & -\frac{1}{3} \sigma_u & 0 \\ 0 & 0 & -\frac{1}{3} \sigma_u \end{bmatrix}. $$
Finally, the equivalent uniaxial stress (or effective stress) is obtained,

$$\bar{\sigma} = \sqrt{\frac{3}{2} \sigma_{ij} \sigma_{ji}} = \sqrt{\frac{3}{2} \sigma_u (\frac{4}{9} + \frac{1}{9} + \frac{1}{9})} = \sqrt{\frac{3}{2} \frac{2}{3} |\sigma_u|} = |\sigma_u| \implies \bar{\sigma} = |\sigma_u|$$

8.3 Principal Stress Space

Consider a system of Cartesian axes in $\mathbb{R}^3 \{ x = \sigma_1, y = \sigma_2, z = \sigma_3 \}$ such that each stress state, characterized by the values of the three principal stresses $\sigma_1 \geq \sigma_2 \geq \sigma_3$, corresponds to a point in this space, which is known as the principal stress space.\footnote{The principal stress space is also known as the Haigh-Westergaard stress space.}

Definition 8.1. The hydrostatic stress axis is the locus of points in the principal stress space that verify the condition $\sigma_1 = \sigma_2 = \sigma_3$ (see Figure 8.3). The points located on the hydrostatic stress axis represent hydrostatic states of stress (see Chapter 4, Section 4.4.5).

![Figure 8.3: The principal stress space.](image)
**Definition 8.2.** The *octahedral plane* \( \Pi \) is any of the planes that are perpendicular to the hydrostatic stress axis (see Figure 8.4). The equation of an octahedral plane is

\[
\sigma_1 + \sigma_2 + \sigma_3 = \text{const.}
\]

and the unit normal vector of said plane is

\[
n \equiv \frac{1}{\sqrt{3}} [1, 1, 1]^T.
\]

**8.3.1 Normal and Shear Octahedral Stresses**

Consider \( P \) is a point in the principal stress space with coordinates \( (\sigma_1, \sigma_2, \sigma_3) \). The position vector of this point is defined as \( OP \equiv [\sigma_1, \sigma_2, \sigma_3]^T \) (see Figure 8.5). Now, the octahedral plane \( \Pi \) containing point \( P \) is considered. The intersection of the hydrostatic stress axis with said plane defines point \( A \).

**Definition 8.3.** Based on Figure 8.5, the *normal octahedral stress* is defined as

\[
|OA| = \sqrt{3} \sigma_{\text{oct}}
\]

and the *shear or tangential octahedral stress* is

\[
|AP| = \sqrt{3} \tau_{\text{oct}}.
\]
Remark 8.4. The normal octahedral stress $\sigma_{oct}$ informs of the distance between the origin $O$ of the principal stress space and the octahedral plane that contains point $P$. The locus of points in the principal stress space with the same value of $\sigma_{oct}$ is the octahedral plane placed at a distance $\sqrt{3}\sigma_{oct}$ of the origin.

The shear octahedral stress $\tau_{oct}$ informs of the distance between point $P$ and the hydrostatic stress axis. It is, thus, a measure of the distance that separates the stress state characterized by point $P$ from a hydrostatic stress state. The locus of points in the principal stress space with the same value of $\tau_{oct}$ is a cylinder whose axis is the hydrostatic stress axis and whose radius is $\sqrt{3}\tau_{oct}$.

The distance $|\overrightarrow{OA}|$ can be computed as the projection of the vector $\overrightarrow{OP}$ on the unit normal vector of the octahedral plane, $\mathbf{n}$,

$$|\overrightarrow{OA}| = \overrightarrow{OP} \cdot \mathbf{n} \equiv \begin{bmatrix} \sigma_1, \sigma_2, \sigma_3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \frac{\sqrt{3}}{3} (\sigma_1 + \sigma_2 + \sigma_3) = \sqrt{3}\sigma_m \Rightarrow$$

$$|\overrightarrow{OA}| = \sqrt{3}\sigma_{oct} \quad (8.15)$$
where the definition (8.9) of mean stress \( \sigma_m \) has been taken into account.

The distance \(|\overline{AP}|\) can be obtained solving for the right triangle \(OAP\) in Figure 8.5,

\[
|\overline{AP}|^2 = OP^2 - OA^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)^2.
\]

(8.17)

By means of several algebraic operations, this distance can be expressed in terms of the second invariant of the deviatoric stress tensor in (8.14), \( J'_2 \), as

\[
|\overline{AP}| = \sqrt{2}J'_2 \quad \Rightarrow \quad |\overline{AP}| = \sqrt{2}(J'_2)^{1/2}
\]

(8.18)

\[
|\overline{AP}| = \sqrt{3} \tau_{oct}
\]

(8.19)

Alternative expressions of \( \tau_{oct} \) in terms of the value of \( J'_2 \) in (8.14) are

\[
\tau_{oct} = \frac{1}{\sqrt{3}} \left( \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)^2 \right)^{1/2}
\]

and

\[
\tau_{oct} = \frac{1}{3\sqrt{3}} \left( (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 \right)^{1/2}.
\]

(8.20)

**Remark 8.5.** In a pure spherical stress state of \( \sigma \),

\[
\sigma = \sigma_{sph} \iff \sigma' = \sigma - \sigma_{sph} = 0 \iff J'_2 = 0 \iff \tau_{oct} = 0.
\]

A spherical stress state is characterized by \( \tau_{oct} = 0 \) and, thus, is located on the hydrostatic stress axis (see Figure 8.5).

In a pure deviatoric stress state of \( \sigma \),

\[
\sigma = \sigma' \iff \sigma_m = \text{Tr}(\sigma) = \text{Tr}(\sigma') = 0 \iff \sigma_{oct} = 0.
\]

A deviatoric stress state is characterized by \( \sigma_{oct} = 0 \) and, therefore, is located on the octahedral plane containing the origin of the principal stress space.
Remark 8.6. A point $P$ of the principal stress space is univocally characterized by the three invariants $I_1 \equiv J_1$, $J'_2$ and $J'_3$ (see Figure 8.6):

- The first stress invariant $I_1$ characterizes the distance ($= \sqrt{3} \sigma_{oct}$) from the origin to the octahedral plane $\Pi$ containing this point through the relation $\sigma_{oct} = I_1 / 3$. Thus, it places point $P$ in a certain octahedral plane.

- The second deviatoric stress invariant $J'_2$ characterizes the distance ($= \sqrt{3} \tau_{oct}$) from the hydrostatic stress axis to the point. Thus, it places point $P$ on a certain circle in the octahedral plane with center in the hydrostatic stress axis and radius $\sqrt{3} \tau_{oct} = \sqrt{2} (J'_2)^{1/2}$.

- The third deviatoric stress invariant $J'_3$ characterizes the position of the point on this circle by means of an angle $\theta(J'_3)$.

Figure 8.6: Univocal definition of a point by means of the invariants $I_1$, $J'_2$ and $J'_3$. 
Remark 8.7. Figure 8.7 shows the projection of the principal stress space on an octahedral plane $\Pi$. The division of the stress space into six sectors can be observed in this projection. Each sector is characterized by a different ordering of the principal stresses and the sectors are separated by the projections on the plane of the bisectors $\sigma_2 = \sigma_3$, $\sigma_1 = \sigma_3$ and $\sigma_1 = \sigma_2$.

Selecting the criterion $\sigma_1 \geq \sigma_2 \geq \sigma_3$ automatically reduces the feasible work domain to the sector marked in gray in the figure. The intersection of any surface of the type $f(\sigma_1, \sigma_2, \sigma_3) = 0$ with the plane $\Pi$ is reduced to a curve in said sector.

This curve can be automatically extended to the rest of sectors, that is, the curve obtained with the same function $f(\sigma_1, \sigma_2, \sigma_3) = 0$ but considering the different orderings of the principal stresses can be easily plotted, by considering the symmetry conditions with respect to the bisector planes. The resulting curve presents, thus, three axes of symmetry with respect to each of the axis in Figure 8.7.
8.4 Rheological Models

Rheological models are idealizations of mechanical models, constructed as a combination of simple elements, whose behavior is easily intuitable, and that allow perceiving more complex mechanical behaviors. Here, as a step previous to the analysis of elastoplastic models, frictional rheological models will be used to introduce the concept of irrecoverable or permanent strain and its consequences.

8.4.1 Elastic Element (Spring Element)

The elastic rheological model is defined by a spring with constant $E$ (see Figure 8.8). The model establishes a proportionality between stress and strain, both in loading and unloading, being the constant $E$ the proportionality factor (see Figure 8.8).

![Figure 8.8: Stress-strain relation in an elastic model.](image)

8.4.2 Frictional Element

Consider a solid block placed on a rough surface (see Figure 8.9) and subjected to a vertical compressive load $N$ and a horizontal load $F$ (positive rightward and negative leftward). $\delta$ is the horizontal displacement of the block. The Coulomb friction model\(^2\) establishes that the modulus of the reaction force $R$ exerted by the contact surface on the block cannot exceed a certain limit value $F_u = \mu N$, where $\mu \geq 0$ is the friction coefficient between the block and the surface. Consequently, while the load $F$ is below said limit value, the block does not move. When the limit value $F_u = \mu N$ is reached, the block starts moving in a quasi-static state (without any acceleration). To maintain the quasi-static regime, this limit value must not be exceeded. These concepts can be mathematically expressed as

\(^2\) The Coulomb friction model is also known as dry friction model.

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\[ |F| < \mu N \iff \delta = 0 \text{ there is no motion}, \]
\[ |F| = \mu N \iff \delta \neq 0 \text{ there is motion}, \]
\[ |F| > \mu N \text{ impossible}. \] (8.21)

The behavior of the Coulomb friction model, in terms of the force-displacement relation \( F - \delta \), is graphically represented in Figure 8.9, both for positive values of the load \( F \) (rightward motion) and negative ones (leftward motion).

By analogy with the mechanical friction model, the frictional rheological model in Figure 8.10 is defined, where \( \sigma \) is the stress (analogous to the load \( F \) in the Coulomb model) that acts on the device and \( \varepsilon \) is the strain suffered by this device (analogous to the displacement \( \delta \)). This rheological model includes a frictional device characterized by a limit value \( \sigma_e \) (analogous to the role of \( \mu N \) in the Coulomb model) whose value cannot be exceeded.

Figure 8.11 shows the stress-strain curve corresponding to the frictional rheological model for a loading-unloading-reloading cycle, which can be split into the following sections.

---

Figure 8.9: Coulomb's law of friction.

Figure 8.10: Frictional rheological model.
Section 0 - 1: The (tensile) stress $\sigma$ increases until the threshold value $\sigma = \sigma_e$ is reached. There is no strain.

Section 1 - 2: Once the threshold $\sigma = \sigma_e$ has been reached, stress cannot continue increasing although it can keep its value constant. Then, the frictional element flows, generating a strain $\varepsilon$ that grows indefinitely while the stress is maintained (loading process).

Section 2 - 3: At point 2, the tendency of the stress is inverted, stress starts decreasing ($\Delta \sigma < 0$) and unloading begins ($\sigma < \sigma_e$). Further strain increase is automatically halted ($\Delta \varepsilon = 0$). This situation is maintained until stress is canceled ($\sigma = 0$) at point 3. Note that, if the process was to be halted at this point, the initial state of null stress would be recovered but not the initial state of null strain. Instead, a permanent or residual strain would be observed ($\varepsilon \neq 0$). This reveals that, in this model, the trajectory of the stress-strain curve is different in the loading and unloading regimes and that the deformation process is (from a thermodynamic point of view) irreversible in character.

Section 3 - 4: Beyond point 3, the sign of the stress is inverted and stress becomes compressive. However, since $|\sigma| < \sigma_e$, no changes in strain are observed ($\Delta \varepsilon = 0$).

Section 4 - 5: At point 4, the criterion $|\sigma| = \sigma_e$ is satisfied and the model enters a loading regime again. The element flows at a constant stress value $\sigma = -\sigma_e$, generating negative strain ($\Delta \varepsilon < 0$), which progressively reduces the accumulated strain. Finally, at point 5, the initial strain state is recovered, but not the original stress state. Beyond this point, if unloading was imposed, there would be a corresponding decrease in stress until the cycle was closed at point 0. Conversely, the loading regime could continue, generating a permanent negative strain.

### 8.4.3 Elastic-Frictional Model

The basic rheological elements, elastic and frictional, can be combined to produce a more complex model, named elastic-frictional model, by placing an elastic element, characterized by the parameter $E$, in series with a frictional element, characterized by the parameter $\sigma_e$ (denoted as elastic limit), as shown in Figure 8.12. Consider $\sigma$ is the stress that acts on the model and $\varepsilon$ is the total strain of this model. Since the basic elements are placed in series, the same
stress will act on both of them. On the other hand, the total strain can be decomposed into the sum of the strain experienced by the elastic element \( \varepsilon^e \) plus the strain experienced by the frictional device \( \varepsilon^f \). The same logic can be applied at incremental level.

\[
\begin{align*}
\sigma &= \sigma^e = \sigma^f \\
\varepsilon &= \varepsilon^e + \varepsilon^f = \frac{\sigma}{E} + \varepsilon^f \\
\Delta \varepsilon &= \Delta \varepsilon^e + \Delta \varepsilon^f
\end{align*}
\]

Additive decomposition of strain \( (8.22) \)

The frictional element does not deform for stresses \( |\sigma| < \sigma_e \), therefore all strains are absorbed by the elastic element.

\[
\begin{align*}
|\sigma| &< \sigma_e \quad \Rightarrow \quad \Delta \varepsilon^f = 0 \quad \Rightarrow \quad \Delta \varepsilon = \Delta \varepsilon^e \\
\Delta \sigma &= E \Delta \varepsilon
\end{align*}
\]

All strain increments are absorbed by the frictional element with a null increment of stress.
This is incompatible with the characteristics of the frictional element.

Figure 8.13 shows the stress-strain curve for a loading-unloading-reloading cycle of the elastic-frictional model, which can be decomposed into the following sections.

**Section 0–1:**

\[ |\sigma| < \sigma_e \implies \Delta \varepsilon^f = 0 \implies \Delta \varepsilon = \Delta \varepsilon^e \]

This section corresponds to the *elastic loading* phase. At the end of the loading, at point 1, the strain is \( \varepsilon = \varepsilon^e = \sigma_e / E \). The value of \( \sigma_e \) at the end of this elastic section justifies its denomination as *elastic limit*.

**Section 1–2:**

\[ |\sigma| = \sigma_e \implies \Delta \varepsilon^f \neq 0 \implies \begin{cases} \varepsilon = \frac{\sigma_e}{E} + \varepsilon^f \\ \Delta \varepsilon = \Delta \varepsilon^f > 0 \end{cases} \]

This section corresponds to the *frictional loading* during which no deformation is generated in the elastic element (*no elastic strain is generated*) and all increments of strain are absorbed by the frictional element.

**Section 2–3:**

\[ |\sigma| < \sigma_e \implies \Delta \varepsilon^f = 0 \implies \Delta \varepsilon = \Delta \varepsilon^e \]

This section corresponds to the *elastic unloading*. At the end of the unloading, at point 3, the initial state of null stress is recovered (\( \sigma = 0 \)). Consequently, the elastic strain at this point is \( \varepsilon = \varepsilon^e = \sigma_e / E \) and, thus, the residual or irrecoverable strain is \( \varepsilon = \varepsilon^f \neq 0 \). That is, the strain generated by the frictional element during the frictional loading section 1–2 is not recovered during this phase of stress relaxation to zero. This allows qualifying the *frictional component of strain \( \varepsilon^f \) as an irrecoverable or irreversible strain*.

**Section 3–4:**

\[ |\sigma| < \sigma_e \implies \Delta \varepsilon^f = 0 \implies \Delta \varepsilon = \Delta \varepsilon^e \]

This section corresponds to the *elastic reloading* phase, similar to section 0–1 but with a compressive stress (\( \sigma < 0 \)). The frictional component of strain is not modified during the reloading and the final value, at point 4, of the elastic strain is \( \varepsilon^e = -\sigma_e / E \).

**Section 4–5:**

\[ |\sigma| = \sigma_e \implies \Delta \varepsilon^f \neq 0 \implies \begin{cases} \varepsilon = -\frac{\sigma_e}{E} + \varepsilon^f \\ \Delta \varepsilon = \Delta \varepsilon^f < 0 \end{cases} \]
This section corresponds to the frictional reloading during which negative frictional strain is generated ($\Delta \varepsilon^f < 0$). Therefore, the total value of the frictional strain decreases until it becomes zero at point 5 (characterized by $\varepsilon = \varepsilon^e = -\sigma_e/E$ and $\varepsilon^f = 0$). An additional elastic unloading at this point would result in recovering the initial state 0.

8.4.4 Frictional Model with Hardening

Consider the rheological model in Figure 8.14 composed of an elastic element (characterized by the parameter $H'$, which will be denoted as hardening modulus) and a frictional element (characterized by the elastic limit $\sigma_e$) placed in parallel. The parallel arrangement results in both rheological elements sharing the same strain, while the total stress in the model is the sum of the stress in the frictional element ($\sigma^{(1)}$) plus the stress in the elastic element ($\sigma^{(2)}$).

\[
\begin{align*}
\sigma &= \sigma^{(1)} + \sigma^{(2)} \\
\Delta \sigma &= \Delta \sigma^{(1)} + \Delta \sigma^{(2)} \\
\varepsilon &= \varepsilon^e = \varepsilon^f
\end{align*}
\]  

(8.23)

Figure 8.14: Frictional model with hardening.
Analyzing separately the behavior of each element results in:

\( a) \) Frictional element

\[
\begin{align*}
|\sigma^{(1)}| < \sigma_e & \quad \Delta \varepsilon^f = \Delta \varepsilon = 0 \\
|\sigma^{(1)}| = \sigma_e & \quad \Delta \varepsilon^f = \Delta \varepsilon \neq 0 \\
|\sigma^{(1)}| > \sigma_e & \quad \text{impossible}
\end{align*}
\]

(8.24)

\( b) \) Elastic element

\[
\begin{align*}
\left\{ \begin{array}{c}
\sigma^{(2)} = H' \varepsilon^e = H' \varepsilon \\
\Delta \sigma^{(2)} = H' \Delta \varepsilon^e = H' \Delta \varepsilon
\end{array} \right.
\]

(8.25)

\( c) \) Combining (8.24) and (8.25) leads to

\[
|\sigma^{(1)}| = |\sigma - \sigma^{(2)}| = |\sigma - H' \varepsilon|
\]

(8.26)

In agreement with (8.24) and (8.25), the following situations can be established regarding the rheological model:

\[ \begin{align*}
|\sigma^{(1)}| < \sigma_e & \iff |\sigma - H' \varepsilon| < \sigma_e \\
\Delta \varepsilon^f = \Delta \varepsilon = 0 \\
\Delta \sigma^{(2)} = H' \Delta \varepsilon^e = H' \Delta \varepsilon = 0
\end{align*} \]

All the stress is absorbed by the frictional device and strain is null.

\[ \begin{align*}
|\sigma^{(1)}| = \sigma_e & \iff |\sigma - H' \varepsilon| = \sigma_e \\
\Rightarrow \begin{cases}
|\sigma^{(1)}| = \sigma_e \\
\sigma^{(2)} = |\sigma - \sigma^{(1)}|
\end{cases}
\end{align*} \]

\[ \Delta \sigma^{(2)} = \Delta \sigma = H' \Delta \varepsilon \]

All stress increments are totally absorbed by the elastic element.

Figure 8.15 shows the stress-strain curve corresponding to this rheological model for a loading-unloading-reloading cycle, which can be decomposed into the following sections.
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Section 0 – 1:

\[
\left| \sigma^{(1)} \right| < \sigma_e \implies \Delta \varepsilon = 0 \implies \begin{cases} \Delta \sigma^{(2)} = E \Delta \varepsilon = 0 \\ \Delta \sigma^{(1)} = \Delta \sigma \end{cases}
\]

In this section all the stress is absorbed by the frictional element. At the end of the section, at point 1, the strain is \( \varepsilon = 0 \) and the stress is \( \sigma = \sigma_e \). This section is characterized by the condition

\[
\left| \sigma - H' \varepsilon \right| < \sigma_e .
\]

Section 1 – 2:

\[
\left| \sigma^{(1)} \right| = \sigma_e \implies \begin{cases} \sigma = \sigma_e + \sigma^{(2)} \\ \Delta \sigma = \Delta \sigma^{(2)} = H' \Delta \varepsilon \end{cases}
\]

This is a loading section in which all stress is absorbed by the elastic element. In global terms, the model increases its capacity to resist stress (the model is said to suffer hardening) proportionally to the increment of strain, being the proportionality factor the hardening modulus \( H' \). This section is characterized by the condition

\[
\left| \sigma - H' \varepsilon \right| = \sigma_e .
\]

Section 2 – 3:

\[
\left| \sigma^{(1)} \right| < \sigma_e \implies \Delta \varepsilon = 0 \implies \begin{cases} \Delta \sigma^{(1)} = \Delta \sigma \\ \Delta \sigma^{(2)} = 0 \end{cases}
\]

In this section the stress in the frictional element decreases with a null increment of strain and keeping the stress constant in the elastic element. This state is maintained until stress is totally inverted in the frictional element. Thus, at point 3, the stress is \( \sigma^{(1)} = -\sigma_e \). This section is characterized by the condition

\[
\left| \sigma - H' \varepsilon \right| < \sigma_e .
\]

Section 3 – 4:

\[
\left| \sigma^{(1)} \right| = \sigma_e \implies \begin{cases} \sigma = -\sigma_e + \sigma^{(2)} \\ \Delta \sigma = \Delta \sigma^{(2)} = H' \Delta \varepsilon \end{cases}
\]

The situation is symmetrical with respect to section 1 – 2, with the elastic element decreasing the stress it can bear, until the stress becomes null at point 3,
where $\sigma^{(1)} = -\sigma_e$ and $\sigma^{(2)} = 0$. This section is characterized by the condition

$$|\sigma - H'\varepsilon| = \sigma_e.$$

Beyond this point, relaxation of the stress in the frictional element leads to the original state at point 0.

Figure 8.15: Stress-strain curve for a loading-unloading-reloading cycle in a frictional rheological model with hardening.

### 8.4.5 Elastic-Frictional Model with Hardening

Combining now an elastic element, with elastic modulus $E$, in series with the frictional model introduced in section 8.4.4, which has a hardening modulus $H'$ and an elastic limit $\sigma_e$, the elastic-frictional model with hardening shown in Figure 8.16 is obtained.

Applying the stress equilibrium and strain compatibility equations on the model (see Figure 8.16) results in

$$\begin{align*}
\varepsilon &= \varepsilon^e + \varepsilon^f \\
\Delta \varepsilon &= \Delta \varepsilon^e + \Delta \varepsilon^f \\
\sigma &= \sigma^e = \sigma^f \\
\Delta \sigma &= \Delta \sigma^e = \Delta \sigma^f
\end{align*}$$

(8.27)

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where $\sigma^e$ and $\sigma^f$ represent, respectively, the stresses sustained by the elastic element and the frictional model with hardening. Combining now the behavior of an elastic element (see Figure 8.8) with that of the frictional model with hardening in Figure 8.14, yields the following situations:

- $|\sigma - H'\varepsilon_f| < \sigma_e \implies \begin{cases} \Delta \varepsilon_f = 0 \\ \Delta \sigma = \Delta \varepsilon_e \implies \Delta \sigma = E \Delta \varepsilon \end{cases}$

  The frictional element with hardening does not deform and the increment of strain $\Delta \varepsilon$ is completely absorbed by the elastic element. This case is denoted as elastic process.

- $|\sigma - H'\varepsilon_f| = \sigma_e$

  \[ \sigma \Delta \sigma > 0 \iff \begin{cases} \sigma > 0 \text{ and } \Delta \sigma > 0 \\ \sigma < 0 \text{ and } \Delta \sigma < 0 \end{cases} \implies \begin{cases} \Delta \sigma = \Delta \varepsilon_e + \Delta \varepsilon_f = \frac{1}{E} \Delta \sigma + \frac{1}{H'} \Delta \sigma = \frac{E + H'}{EH'} \Delta \sigma \\ \Delta \varepsilon = \frac{E' \Delta \varepsilon}{E + H'} \end{cases} \]

  The strain increment is absorbed by the two elements of the model (the frictional one with hardening and the elastic one). The relation between the stress increment $\Delta \sigma$ and the strain increment $\Delta \varepsilon$ is given by the
elastico-frictional tangent modulus $E^{ef}$. This case is called inelastic loading process.

$$b) \quad \sigma \Delta \sigma < 0 \iff \begin{cases} \sigma > 0 \text{ and } \Delta \sigma < 0 \\ \sigma < 0 \text{ and } \Delta \sigma > 0 \end{cases} \implies \Delta \varepsilon^f = 0 \implies \Delta \varepsilon = \Delta \varepsilon^e \implies \Delta \sigma = E \Delta \varepsilon$$

Every strain increment $\Delta \varepsilon$ is absorbed by the elastic element. This case is named elastic unloading process.

Figure 8.17 shows the stress-strain curve corresponding to the model for a loading-unloading-reloading cycle, in which the following sections can be differentiated.

**Section 0 – 1** and **section 2 – 3:**

$$|\sigma - H' \varepsilon^f| < \sigma_e \implies \Delta \sigma = \sigma \Delta \sigma$$

Correspond to elastic processes.

**Section 1 – 2** and **section 3 – 4:**

$$\begin{cases} |\sigma - H' \varepsilon_f| = \sigma_e \\ \sigma \Delta \sigma > 0 \end{cases} \implies \Delta \sigma = E^{ef} \Delta \varepsilon$$

Correspond to inelastic loading processes.

**Point 2:**

$$\begin{cases} |\sigma - H' \varepsilon_f| = \sigma_e \\ \sigma \Delta \sigma < 0 \end{cases} \implies \Delta \sigma = E \Delta \varepsilon$$

Corresponds to an elastic unloading process.

Note that if $H' = 0$, then $E^{ef} = 0$, and the elastic-frictional model in Figure 8.13 is recovered.
8.5 Elastoplastic Phenomenological Behavior

Consider a steel bar of length $\ell$ and cross-section $A$ subjected to a tensile force $F$ at its extremes. The stress in the bar will be $\sigma = F/A$ (see Figure 8.18) and the corresponding strain can be estimated as $\varepsilon = \delta/\ell$, where $\delta$ is the lengthening of the bar. If the bar is subjected to several loading and unloading cycles, the response typically obtained, in terms of stress-strain curve $\sigma - \varepsilon$, is as indicated in Figure 8.19.

Observation of the first cycle reveals that, as long as the stress does not exceed the value $\sigma_e$ (denoted as elastic limit) in point 1, the behavior is linear elastic, characterized by the elastic modulus $E$ ($\sigma = E\varepsilon$), and there do not exist irrecoverable strains (in a possible posterior unloading, the strain produced during loading would be recovered).

For stress values above $\sigma_e$, the behavior ceases to be elastic and part of the strain is no longer recovered during an ensuing unloading to null stress (point 3). There appears, thus, a remaining strain named plastic strain, $\varepsilon_p$. However, during the unloading section 2 − 3 the behavior is again, in an approximate form, incrementally elastic ($\Delta\sigma = E\Delta\varepsilon$). The same occurs with the posterior reloading 3 − 2, which produces an incrementally elastic behavior, until the stress reaches, in point 2, the maximum value it will have achieved during the loading.
process. From this point on, the behavior is no longer incrementally elastic (as if the material *remembered* the maximum stress to which it has been previously subjected). A posterior loading-unloading-reloading cycle $2 - 4 - 5 - 4$ exposes again that, during section $2 - 4$, additional plastic strain is generated, which appears in the form of permanent strain in point 5, and, also, additional elastic strain $\varepsilon^e$ is produced, understood as the part of the strain that can be recovered during the unloading section $4 - 5$.

### 8.5.1 Bauschinger Effect

Consider a sample of virgin material (a material that has not suffered previous states of inelastic strain) subjected to an uniaxial tensile test and another sample of the same virgin material subjected to an uniaxial compressive test. In certain materials, the responses obtained, in terms of the stress-strain curve $\sigma - \varepsilon$ in Figure 8.20, for both tests are symmetrical with respect to the origin. That is, in the tensile test the response is elastic up to a value of $\sigma = \sigma_e$ (*tensile elastic limit*) and in the compressive response the answer is also elastic up to a value of $\sigma = -\sigma_e$ (*compressive elastic limit*), being the rest of both curves (for an assumed regime of monotonous loading) also symmetrical. In this case, the stress-strain curve of the virgin material is said to be symmetrical in tension and compression.

Suppose now that a specimen that has been *previously subjected to a history of plastic strains*\(^3\), for example a tensile loading-unloading cycle such as the $0 - 1 - 2 - 3$ cycle shown in Figure 8.19, undergoes now a compressive test. Consider also that $\sigma_y > \sigma_e$ is the maximum stress the material has been subjected to.

---

\(^3\) This procedure is known as *cold hardening* and its purpose is to obtain an apparent elastic limit that is superior to that of the virgin material $\sigma_y > \sigma_e$. 

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subjected to during the loading process. An hypothetical symmetrical behavior would result in the material having now an elastic behavior in the stress range $[-\sigma_y, \sigma_y]$. However, in certain cases, the elastic behavior in compression ends much earlier (see Figure 8.20). This is the effect known as Bauschinger effect or kinematic hardening. Note that the stress-strain curve of the elastic-frictional model in Figure 8.17 exhibits this type of hardening.

![stress-strain curve](image)

**Remark 8.8.** In view of the phenomenological behavior observed in Figure 8.19 and in Figure 8.20, the elastoplastic behavior is characterized by the following facts:

1) Unlike in the elastic case, there does not exist unicity in the stress-strain relation. A same value of strain can correspond to infinite values of stress and vice-versa. The stress value depends not only on the strain, but also on the loading history.

2) There does not exist a linear relation between stress and strain. At most, this linearity may be incremental in certain sections of the deformation process.

3) Irrecoverable or irreversible strains are produced in a loading-unloading cycle.
8.6 Incremental Theory of Plasticity in 1 Dimension

The elastoplastic behavior analyzed in section 8.5 can be modeled using mathematical models of certain complexity. One of the most popular approximations is the so-called incremental theory of plasticity. In a one-dimensional case, the theory seeks to approximate a stress-strain behavior such as the one observed in Figure 8.19 by means of piece-wise approximations using elastic and inelastic regions such as the ones shown in Figure 8.21. The generalization to several dimensions requires the introduction of more abstract concepts.

8.6.1 Additive Decomposition of Strain. Hardening Variable

The total strain \( \varepsilon \) is decomposed into the sum of an elastic (or recoverable) strain \( \varepsilon^e \), governed by Hooke’s law, and a plastic (or irrecoverable) strain \( \varepsilon^p \),

\[
\begin{align*}
\varepsilon &= \varepsilon^e + \varepsilon^p \\
\varepsilon^e &= \frac{\sigma}{E} \\
\varepsilon^p &= \frac{d\sigma}{E}
\end{align*}
\]

(8.28)

where \( E \) is the elastic modulus. In addition, the hardening variable \( \alpha (\sigma, \varepsilon^p) \) is defined by means of the evolution equation as follows:

\[
\begin{align*}
d\alpha &= \text{sign}(\sigma) d\varepsilon^p \\
d\alpha &\geq 0 \\
\alpha |_{\varepsilon^p=0} &= 0
\end{align*}
\]

(8.29)

---

4 Up to a certain point, these models may be inspired, albeit with certain limitations, in the elastic-frictional rheological models described in section 8.4.

5 Here, the sign operator is used, which is defined as \( x \geq 0 \iff \text{sign}(x) = +1 \) and \( x < 0 \iff \text{sign}(x) = -1 \).
Remark 8.9. Note that the hardening variable \( \alpha \) is always positive, in agreement with its definition in (8.29), and, considering the modules of the expression \( d\alpha = \text{sign}(\sigma) d\varepsilon^p \), results in

\[
\begin{align*}
    d\alpha &= |d\alpha| = |\text{sign}(\sigma)| |d\varepsilon^p| \implies d\alpha = |d\varepsilon^p| \\
    &= 1
\end{align*}
\]

Then, for a process with monotonously increasing plastic strains, both variables coincide,

\[
    d\varepsilon^p \geq 0 \implies \alpha = \int_0^{\varepsilon^p} |d\varepsilon^p| = \int_0^{\varepsilon^p} d\varepsilon^p = \varepsilon^p.
\]

However, if the process does not involve a monotonous increase, the plastic strain may decrease and its value no longer coincides with that of the hardening variable \( \alpha \).

### 8.6.2 Elastic Domain. Yield Function. Yield Surface

The elastic domain in the stress space is defined as the interior of the domain enclosed by the surface \( F(\sigma, \alpha) = 0 \),

\[
\text{Elastic domain: } E_\sigma := \{ \sigma \in \mathbb{R} \mid F(\sigma, \alpha) < 0 \}
\]

where the function \( F(\sigma, \alpha) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \) is denoted as yield function.

The initial elastic domain \( E_\sigma^0 \) is defined as the elastic domain corresponding to a null plastic strain \( (\varepsilon^p = \alpha = 0) \),

\[
\text{Initial elastic domain: } E_\sigma^0 := \{ \sigma \in \mathbb{R} \mid F(\sigma, 0) < 0 \}
\]

An additional requirement of the initial elastic domain is that it must contain the null stress state,

\[
0 \in E_\sigma^0 \implies F(0, 0) < 0,
\]

and this is achieved by defining a yield function of the type
Yield function: \( F(\sigma, \alpha) \equiv |\sigma| - \sigma_y(\alpha) \) \hspace{1cm} (8.33)

where \( \sigma_y(\alpha) > 0 \) is known as the yield stress. The initial value (for \( \alpha = 0 \)) of the yield stress is the elastic limit \( \sigma_e \) (see Figure 8.22) and the function \( \sigma_y(\alpha) : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is named hardening law.

The yield surface is defined as the boundary of the elastic domain.

Yield surface: \( \partial \mathbb{E}_\sigma := \{ \sigma \in \mathbb{R} | F(\sigma, \alpha) \equiv |\sigma| - \sigma_y(\alpha) = 0 \} \) \hspace{1cm} (8.34)

The elastic domain \( \mathbb{E}_\sigma \) together with its boundary \( \partial \mathbb{E}_\sigma \) determine the admissible stress space (domain) \( \bar{\mathbb{E}}_\sigma \)

Admissible stress space:
\[
\bar{\mathbb{E}}_\sigma = \mathbb{E}_\sigma \cup \partial \mathbb{E}_\sigma = \{ \sigma \in \mathbb{R} | F(\sigma, \alpha) \equiv |\sigma| - \sigma_y(\alpha) \leq 0 \}
\] \hspace{1cm} (8.35)

and it is postulated that any feasible (admissible) stress state must belong to the admissible stress space \( \bar{\mathbb{E}}_\sigma \). Considering the definitions of elastic domain in (8.30), yield surface in (8.34) and admissible stress space in (8.35), the following is established.
\[ F(\sigma, \alpha) < 0 \iff |\sigma| < \sigma_y(\alpha) \iff \begin{cases} \sigma \text{ in the elastic domain} \\ (\sigma \in E_\sigma) \end{cases} \]

\[ F(\sigma, \alpha) = 0 \iff |\sigma| = \sigma_y(\alpha) \iff \begin{cases} \sigma \text{ on the yield surface} \\ (\sigma \in \partial E_\sigma) \end{cases} \]

\[ F(\sigma, \alpha) > 0 \iff |\sigma| > \sigma_y(\alpha) \iff \text{non-admissible stress state} \]

\( (8.36) \)

\[ \text{Remark 8.10.} \text{ Note how, in (8.35), the admissible stress space depends on the hardening variable } \alpha. \text{ The admissible domain evolves with the yield function } \sigma_y(\alpha) \text{ such that (see Figure 8.22)} \]

\[ \bar{E}_\sigma \equiv [-\sigma_y(\alpha), \sigma_y(\alpha)] \]

### 8.6.3 Constitutive Equation

To characterize the response of the material, the following situations are defined:

- **Elastic regime**
  \[ \sigma \in E_\sigma \iff d\sigma = E d\varepsilon \]  
  \( (8.37) \)

- **Elastoplastic regime in unloading**
  \[ \sigma \in \partial E_\sigma \]
  \[ dF(\sigma, \alpha) < 0 \]
  \[ \implies d\sigma = E d\varepsilon \]  
  \( (8.38) \)

- **Elastoplastic regime in plastic loading**
  \[ \sigma \in \partial E_\sigma \]
  \[ dF(\sigma, \alpha) = 0 \]
  \[ \implies d\sigma = E^{ep} d\varepsilon \]  
  \( (8.39) \)

where \( E^{ep} \) is denoted as *elastoplastic tangent modulus*. 

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Remark 8.11. The situation \( \sigma \in \partial E_\sigma \) and \( dF (\sigma, \alpha) > 0 \) cannot occur since, if \( \sigma \in \partial E_\sigma \), from (8.34) results
\[
F (\sigma, \alpha) \equiv |\sigma| - \sigma_y (\alpha) = 0.
\]
If, in addition, \( dF (\sigma, \alpha) > 0 \) then,
\[
F (\sigma + d\sigma, \alpha + d\alpha) = F (\sigma, \alpha) + dF (\sigma, \alpha) > 0
\]
and, in agreement with (8.36), the stress state \( \sigma + d\sigma \) is not admissible.

8.6.4 Hardening Law. Hardening Parameter

The hardening law provides the evolution of the yield stress \( \sigma_y (\alpha) \) in terms of the hardening variable \( \alpha \) (see Figure 8.22). Even though the aforementioned hardening law may be of a more general nature, it is common (and often sufficient) to consider a \emph{linear} hardening law of the type
\[
\sigma_y = \sigma_e + H' \alpha \Rightarrow d\sigma_y (\alpha) = H' d\alpha.
\]
where \( H' \) is known as the \emph{hardening parameter}.

8.6.5 Elastoplastic Tangent Modulus

The value of the elastoplastic tangent modulus \( E^{ep} \) introduced in (8.39) is calculated in the following manner. Consider an elastoplastic regime in plastic loading. Then, from (8.39)\(^6\),
\[
\sigma \in \partial E_\sigma \quad \Rightarrow \quad F (\sigma, \alpha) \equiv |\sigma| - \sigma_y (\alpha) = 0 \quad \Rightarrow \quad \text{(8.41)}
\]
where (8.40) has been taken into account. Introducing the first expression of (8.29) in (8.41) yields
\[
\text{sign} (\sigma) d\sigma - H' \text{sign} (\sigma) de^p = 0 \quad \Rightarrow \quad \text{(8.42)}
\]
\(^6\) The property \( d|x|/dx = \text{sign} (x) \) is used here.
Consider now the additive decomposition of strain defined in (8.28), which together with (8.42) results in

\[ d\varepsilon = d\varepsilon^e + d\varepsilon^p \]

\[ d\varepsilon^e = \frac{1}{E} d\sigma \]

\[ d\varepsilon^p = \frac{1}{H'} d\sigma \]

\[ \Rightarrow d\sigma = \frac{1}{E + \frac{1}{H'}} d\varepsilon \]

\[ \Rightarrow \begin{cases} 
  d\sigma = E_{ep} d\varepsilon \\
  E_{ep} = E \frac{H'}{E + H'}. 
\end{cases} \] (8.43)

### 8.6.6 Uniaxial Stress-Strain Curve

The constitutive equation defined by expressions (8.37) to (8.39) allows obtaining the corresponding stress-strain curve for an uniaxial process of loading-unloading-reloading (see Figure 8.23) in which the following sections are observed.

**Section 0 – 1:**

\[ |\sigma| < \sigma_e \Rightarrow \sigma \in \mathbb{E}_\sigma \Rightarrow \text{Elastic regime} \]

From (8.37), \( d\sigma = Ed\varepsilon \) and the behavior is linear elastic, defining an elastic region in the stress-strain curve.

**Section 1 – 2 – 4:**

\[ F(\sigma, \alpha) \equiv |\sigma| - \sigma_y(\alpha) = 0 \Rightarrow \sigma \in \partial \mathbb{E}_\sigma \]

\[ dF(\sigma, \alpha) = 0 \]

\[ \Rightarrow \text{Elastoplastic regime in plastic loading} \]

From (8.39), \( d\sigma = E_{ep} d\varepsilon \), defining an elastoplastic region.

**Section 2 – 3 – 2:**

\[ F(\sigma, \alpha) \equiv |\sigma| - \sigma_y(\alpha) < 0 \Rightarrow \sigma \in \partial \mathbb{E}_\sigma \Rightarrow \text{Elastic regime} \]

From (8.37), \( d\sigma = Ed\varepsilon \) and the behavior is linear elastic, defining an elastic region in the stress-strain curve.
Remark 8.12. In point 2 of Figure 8.23 the following two processes are distinguished:

\[
F(\sigma, \alpha) \equiv |\sigma - \sigma_p(\alpha)| = 0 \implies \sigma \in \partial E_\sigma
\]

\[
dF(\sigma, \alpha) < 0
\]

Elastic unloading in section 2 - 3

\[
F(\sigma, \alpha) \equiv |\sigma| - \sigma_p(\alpha) = 0 \implies \sigma \in \partial E_\sigma
\]

\[
dF(\sigma, \alpha) = 0
\]

Plastic loading in section 2 - 4

Remark 8.13. Note that plastic strain is only generated during the plastic loading process in the elastoplastic region (see Figure 8.24).
Remark 8.14. Note the similarity between the stress-strain curve in Figure 8.23 and the one obtained with the elastic-frictional rheological model with hardening in section 8.4.5 (Figure 8.17). The friction strain in said model is equivalent to the plastic strain in the incremental theory of plasticity.

Remark 8.15. The hardening parameter $H'$ plays a fundamental role in the definition of the slope $E^p$ of the elastoplastic region. Following (8.43),

$$E^p = E \frac{H'}{E + H'}$$

and, depending on the value of $H'$, different situations arise (see Figure 8.25):

- $H' > 0 \Rightarrow E^p > 0 \rightarrow$ Plasticity with strain hardening. The limit case $H' = \infty \Rightarrow E^p = E$ recovers the linear elastic behavior.
- $H' = 0 \Rightarrow E^p = 0 \rightarrow$ Perfect plasticity.
- $H' < 0 \Rightarrow E^p < 0 \rightarrow$ Plasticity with strain softening. The limit case corresponds to $H' = -E \Rightarrow E^p = -\infty$.

Plasticity with strain softening presents a specific problematic regarding the uniqueness of the solution to the elastoplastic problem, which is beyond the scope of this text.
8.7 Plasticity in 3 Dimensions

The incremental theory of plasticity developed in one dimension in section 8.6 can be generalized to a multiaxial stress state (three dimensions) using the same ingredients, that is:

1) **Additive decomposition of strain**

\[
\begin{align*}
\mathbf{e} &= \mathbf{e}^e + \mathbf{e}^p \\
\mathbf{e}^e &= \mathbf{C}^{-1} : \mathbf{\sigma} \\
d\mathbf{e}^{ep} &= \lambda \cdot \frac{\partial G(\mathbf{\sigma}, \alpha)}{\partial \mathbf{\sigma}} \\
d\alpha &= \lambda \\
\alpha &\in [0, \infty)
\end{align*}
\]  

(8.44)

where \(\mathbf{C}^{-1}\) is now the (constant) constitutive elastic tensor defined in chapter 6.

2) **Hardening variable \(\alpha\) and flow rule (evolution equations)**

\[
\begin{align*}
d\mathbf{e}^{ep} &= \lambda \cdot \frac{\partial G(\mathbf{\sigma}, \alpha)}{\partial \mathbf{\sigma}} \\
d\alpha &= \lambda \\
\alpha &\in [0, \infty)
\end{align*}
\]  

(8.45)

where \(\lambda\) is the plastic multiplier and \(G(\mathbf{\sigma}, \alpha)\) is the plastic potential function.
3) Yield function. Elastic domain and yield surface

Yield function

\[
\begin{align*}
F(\sigma, \alpha) &\equiv \phi(\sigma) - \sigma_y(\alpha) \\
\sigma_y(\alpha) &= \sigma_e + H'\alpha \quad \text{(hardening law)}
\end{align*}
\]

Elastic domain

\[
\mathbb{E}_\sigma := \{ \sigma \mid F(\sigma, \alpha) < 0 \}
\]

Initial elastic domain

\[
\mathbb{E}_\sigma^0 := \{ \sigma \mid F(\sigma, 0) < 0 \}
\]

Yield surface

\[
\partial \mathbb{E}_\sigma := \{ \sigma \mid F(\sigma, \alpha) = 0 \}
\]

Admissible stress state

\[
\bar{\mathbb{E}}_\sigma = \mathbb{E}_\sigma \cup \partial \mathbb{E}_\sigma = \{ \sigma \mid F(\sigma, \alpha) \leq 0 \}
\]

where \(\phi(\sigma) \geq 0\) is denoted as the equivalent uniaxial stress, \(\sigma_e\) is the elastic limit obtained in an uniaxial test of the material (it is a material property) and \(\sigma_y(\alpha)\) is the yield stress. The hardening parameter \(H'\) plays the same role as in the uniaxial case and determines the expansion or contraction of the elastic domain \(\mathbb{E}_\sigma\), in the stress space, as \(\alpha\) grows. Consequently,

\[
\begin{align*}
H' > 0 &\implies \text{Expansion of } \mathbb{E}_\sigma \text{ with } \alpha \implies \text{Plasticity with hardening} \\
H' < 0 &\implies \text{Contraction of } \mathbb{E}_\sigma \text{ with } \alpha \implies \text{Plasticity with softening} \\
H' = 0 &\implies \text{Constant elastic domain} \quad (\mathbb{E}_\sigma = \mathbb{E}_\sigma^0) \implies \text{Perfect plasticity}
\end{align*}
\]

4) Loading-unloading conditions (Karush-Kuhn-Tucker conditions) and consistency condition

\[
\begin{align*}
\text{Loading-unloading conditions} &\implies \lambda \geq 0 ; \quad F(\sigma, \alpha) \leq 0 ; \quad \lambda F(\sigma, \alpha) = 0 \\
\text{Consistency condition} &\implies \text{If } F(\sigma, \alpha) = 0 \implies \lambda F(\sigma, \alpha) = 0
\end{align*}
\]

The loading-unloading conditions and the consistency condition are additional ingredients, with respect to the unidimensional case, which allow obtaining, after certain algebraic manipulation, the plastic multiplier \(\lambda\) introduced in (8.45).
8.7.1 Constitutive Equation

Similarly to the uniaxial case, the following situations are differentiated in relation to the constitutive equation:

- **Elastic regime**

  \[ \sigma \in E_\sigma \implies d\sigma = C : d\varepsilon \]  
  \[ (8.49) \]

- **Elastoplastic regime in unloading**

  \[ \sigma \in \partial E_\sigma \]
  \[ dF(\sigma, \alpha) < 0 \]

  \[ \implies d\sigma = C : d\varepsilon \]  
  \[ (8.50) \]

- **Elastoplastic regime in plastic loading**

  \[ \sigma \in \partial E_\sigma \]
  \[ dF(\sigma, \alpha) = 0 \]

  \[ \implies d\sigma = \mathbf{C}^{ep} : d\varepsilon \]  
  \[ (8.51) \]

where \( \mathbf{C}^{ep} \) is known as the elastoplastic constitutive tensor which, after certain algebraic operations considering (8.44) to (8.48), is defined as

\[
\mathbf{C}^{ep}(\sigma, \alpha) = C \left( \frac{\partial G}{\partial \sigma} \cdot \frac{\partial F}{\partial \sigma} : \mathbf{C} \right) \\
H'' \cdot \frac{\partial F}{\partial \sigma} : \mathbf{C} \cdot \frac{\partial G}{\partial \sigma} + \mathbf{C}^{ijkl} \cdot \frac{\partial F}{\partial \sigma_{pq}} \cdot \frac{\partial G}{\partial \sigma_{rs}} \cdot \mathbf{C}^{rs} \\
i, j, k, l \in \{1, 2, 3\}
\]

\[ (8.52) \]

8.8 Yield Surfaces. Failure Criteria

A fundamental ingredient in the theory of plasticity is the existence of an initial elastic domain \( E_\sigma^0 \) (see Figure 8.26) which can be written as

\[
E_\sigma^0 := \{ \sigma \mid F(\sigma) \equiv \phi(\sigma) - \sigma_e < 0 \}
\]

\[ (8.53) \]
and determines a domain in the stress space delimited by the initial yield surface $\partial E^0_\sigma$,

$$\partial E^0_\sigma := \{ \sigma \mid F(\sigma) \equiv \phi(\sigma) - \sigma_e = 0 \}. \quad (8.54)$$

Given that the initial elastic domain contains the origin of the stress space ($\sigma = 0$), every loading process in any point of the medium will include an elastic regime (as long as the trajectory of the stresses remains inside $E^0_\sigma$, see Figure 8.26) that will end at the instant in which said trajectory reaches the yield surface $\partial E^0_\sigma$. The initial yield surface plays then the role of indicating the instant of failure (understood as the end of the elastic behavior) independently of the possible post-failure (plastic) behavior that initiates beyond this instant. Thus, the importance of the initial yield surface and the interest in formulating the mathematical equations that adequately determine this surface for the different materials of interest in engineering.

With the aim of defining the yield surface independently of the reference system (isotropic material)$^8$, even if formulated in the principal stress space, its mathematical equation is typically defined in terms of the stress invariants,

$$F(\sigma) \equiv F(I_1, J_2^1, J_3^1), \quad (8.55)$$

and, since the criterion $\sigma_1 \geq \sigma_2 \geq \sigma_3$ is adopted, its definition only affects the first sector of the principal stress space and can be automatically extended, due to symmetry conditions (see Remark 8.7), to the rest of sectors in Figure 8.7.

Figure 8.26: Initial elastic domain and initial yield surface.

$^8$ An isotropic elastoplastic behavior is characterized by the fact that the yield surface, understood as an additional ingredient of the constitutive equation, is independent of the reference system.
8.8.1 Von Mises Criterion

In the von Mises criterion the yield surface is defined as

\[
F(\sigma) \equiv \bar{\sigma}(\sigma) - \sigma_e = \sqrt{3J_2'} - \sigma_e = 0 \tag{8.56}
\]

where \( \bar{\sigma}(\sigma) = \sqrt{3J_2} \) is the effective stress (see Remark 8.3). An alternative expression is obtained taking (8.19) and (8.20) and replacing them in (8.56), which produces

\[
F(\sigma) \equiv \frac{1}{\sqrt{2}} \left( (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 \right)^{1/2} - \sigma_e = 0. \tag{8.57}
\]

The graphical representation of the von Mises yield surface is shown in Figure 8.27.

Remark 8.16. Equation (8.56) highlights the dependency of the von Mises yield surface solely on the second stress invariant \( J_2' \). Consequently, all the points of the surface are characterized by the same value of \( J_2' \), which defines a cylinder whose axis is the hydrostatic stress axis.
Remark 8.17. The von Mises criterion is adequate as a failure criterion in metals, in which, typically, hydrostatic stress states (both in tensile and compressive loading) have an elastic behavior and failure is due to the presence of deviatoric stress components.

Example 8.2 – Compute the expression of the von Mises criterion for an uniaxial tensile loading case.

Solution

An uniaxial tensile loading case is characterized by the stress state

\[
\sigma = \begin{bmatrix}
\sigma_u & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

The effective stress is known to be \( \bar{\sigma} = |\sigma_u| \) (see Example 8.1) and, replacing in the expression of the von Mises criterion (8.56), yields

\[
F(\sigma) \equiv \bar{\sigma}(\sigma) - \sigma_e = |\sigma_u| - \sigma_e.
\]

Thus, the initial elastic domain is characterized in the same way as in unidimensional plasticity seen in Section 8.6.2, by the condition

\[
F(\sigma) < 0 \implies |\sigma_u| < \sigma_e.
\]

Example 8.3 – Compute the expression of the von Mises criterion for a stress state representative of a beam under composed flexure.

Solution

The stress state for a beam under composed flexure is
Yield Surfaces. Failure Criteria

\[ \sigma \equiv \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \sigma_m = \frac{1}{3} \sigma_x \\Rightarrow \]

Then, the second stress invariant \( J'_2 \) is computed as

\[ J'_2 = \frac{1}{2} \sigma' \cdot \sigma' = \frac{1}{2} \left( \frac{2}{3} \sigma_x^2 + \tau_{xy} \cdot \tau_{xy} \right) = \frac{1}{3} \sigma_x^2 + \tau_{xy}^2. \]

And the effective stress is obtained for the von Mises criterion,

\[ \tilde{\sigma} = \sqrt{3J'_2} = \sqrt{\sigma_x^2 + 3\tau_{xy}^2} \Rightarrow F(\sigma) < 0 \Rightarrow \tilde{\sigma} < \sigma_e \Rightarrow \]

\[ \sigma_{co} = \sqrt{\sigma_x^2 + 3\tau_{xy}^2} < \sigma_e. \]

The comparison stress, \( \sigma_{co} = \sqrt{\sigma_x^2 + 3\tau_{xy}^2} \), which can be regarded as a scalar for comparison with the uniaxial elastic limit \( \sigma_e \), is commonly used in the design standards of metallic structures.
8.8.2 Tresca Criterion or Maximum Shear Stress Criterion

The Tresca criterion, also known as the maximum shear stress criterion, states that the elastic domain ends, for a certain point in the medium, when the maximum shear stress acting on any of the planes containing this point, $\tau_{\text{max}}$, reaches half the value of the uniaxial elastic limit $\sigma_e$,

$$\tau_{\text{max}} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_e}{2}.$$  \hspace{1cm} (8.58)

Figure 8.28 illustrates the failure situation in terms of Mohr’s circle in three dimensions. In a loading process in which this circle increases starting from the origin, the elastic behavior ends when the circle with radius $\tau_{\text{max}}$ becomes tangent to the straight line $\tau = \tau_{\text{max}} = \sigma_e/2$.

It follows from (8.58) that the Tresca criterion can be written as

$$F(\sigma) \equiv (\sigma_1 - \sigma_3) - \sigma_e = 0$$ \hspace{1cm} (8.59)

Remark 8.18. It can be verified that the Tresca criterion is written in an unequivocal form as a function of $J_2$ and $J_3$ and does not depend on the first stress invariant $I_1$:

$$F(\sigma) \equiv (\sigma_1 - \sigma_3) - \sigma_e \equiv F(J_2^*, J_3^*)$$

Figure 8.28: Representation of the Tresca criterion using Mohr’s circle in three dimensions.
Figure 8.29: Tresca criterion in the principal stress space.

Figure 8.29 shows the yield surface corresponding to the Tresca criterion in the principal stress space, which results in an hexahedral prism whose axis is the hydrostatic stress axis.

**Remark 8.19.** Since the Tresca criterion does not depend on the first stress invariant (and, therefore, on the stress $\sigma_{\text{oct}}$, see (8.16)), the corresponding yield surface does not depend on the distance from the origin to the octahedral plane containing the point (see Remark 8.4). Thus, if a point in the stress space, characterized by its stress invariants $(I_1, J'_2, J'_3)$, is on said yield surface, all the points in the stress space with the same values of $J'_2$ and $J'_3$ will also be on this surface. This circumstance qualifies the yield surface as a prismatic surface whose axis is the hydrostatic stress axis.

On the other hand, the dependency on the two invariants $J'_2$ and $J'_3$, prevents (unlike in the case of the von Mises criterion) the surface from being cylindrical. In short, the symmetry conditions establish that the surface of the Tresca criterion be an hexagonal prism inscribed in the von Mises cylinder (see Figure 8.29).

**Remark 8.20.** The Tresca criterion is used to model the behavior of metals, in a similar manner to the case of the von Mises criterion (see Remark 8.17).
Example 8.4 – Compute the expression of the Tresca criterion for an uniaxial tensile loading case.

Solution

An uniaxial tensile load case is characterized by the stress state

\[ \sigma \equiv \begin{bmatrix} \sigma_u & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \]

For the case \( \sigma_u \geq 0 \),

\[ \sigma_1 = \sigma_u \quad \sigma_3 = 0 \quad \implies \quad F(\sigma_1, \sigma_2, \sigma_3) = (\sigma_1 - \sigma_3) - \sigma_e = \sigma_u - \sigma_e = |\sigma_u| - \sigma_e. \]

For the case \( \sigma_u < 0 \),

\[ \sigma_1 = 0 \quad \sigma_3 = \sigma_u \quad \implies \quad F(\sigma_1, \sigma_2, \sigma_3) = (\sigma_1 - \sigma_3) - \sigma_e = -\sigma_u - \sigma_e = |\sigma_u| - \sigma_e. \]

And the initial elastic domain is then characterized in the same way as in the one-dimensional plasticity seen in Section 8.6.2, by the condition

\[ F(\sigma) < 0 \quad \implies \quad |\sigma_u| < \sigma_e. \]

8.8.3 Mohr-Coulomb Criterion

The Mohr-Coulomb criterion can be viewed as generalization of the Tresca criterion, in which the maximum shear stress sustained depends on the own stress state of the point (see Figure 8.30). The yield line, in the space of Mohr’s circle, is a straight line characterized by the cohesion \( c \) and the internal friction angle \( \phi \), both of which are considered to be material properties,

\[ \tau = c - \sigma \tan \phi. \quad (8.60) \]
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The end of the elastic behavior (failure) in an increasing load process takes place when the first point in the Mohr’s circle (corresponding to a certain plane) reaches the aforementioned yield line. The shear stress in this plane, \( \tau \), becomes smaller as the normal stress \( \sigma \) in the plane increases. It therefore becomes obvious that the behavior of the model under tensile loading is considerably different to the behavior under compressive loading. As can be observed in Figure 8.30, the yield line crosses the normal stress axis in the positive side of these stresses, limiting thus the material’s capacity to withstand tensile loads.

To obtain the mathematical expression of the yield surface, consider a stress state for which plasticization initiates. In such case, the corresponding Mohr’s circle is defined by the major and minor principal stresses and is tangent to the yield line at point \( A \) (see Figure 8.31), verifying

\[
R = \frac{\sigma_1 - \sigma_3}{2} \quad \Rightarrow \quad \begin{cases} 
\tau_A = R \cos \phi = \frac{\sigma_1 - \sigma_3}{2} \cos \phi \\
\sigma_A = \frac{\sigma_1 + \sigma_3}{2} + R \sin \phi = \frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_1 - \sigma_3}{2} \sin \phi 
\end{cases}
\]

and, replacing (8.61) in (8.60), results in

\[
\tau_A = c - \sigma_A \tan \phi \quad \Rightarrow \quad \tau_A + \sigma_A \tan \phi - c = 0 \quad \Rightarrow
\]

\[
\frac{\sigma_1 - \sigma_3}{2} \cos \phi + \left( \frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_1 - \sigma_3}{2} \sin \phi \right) \tan \phi - c = 0 \quad \Rightarrow \quad (8.62)
\]

\[
\left( \sigma_1 - \sigma_3 \right) + \left( \sigma_1 + \sigma_3 \right) \sin \phi - 2c \cos \phi = 0.
\]
Mohr-Coulomb criterion:

\[ F(\sigma) = (\sigma_1 - \sigma_3) + (\sigma_1 + \sigma_3)\sin\phi - 2c\cos\phi = 0 \]  \hspace{1cm} (8.63)

**Remark 8.21.** The equation

\[ F(\sigma) = (\sigma_1 - \sigma_3) + (\sigma_1 + \sigma_3)\sin\phi - 2c\cos\phi = 0, \]

which is linear in \(\sigma_1\) and \(\sigma_3\), defines a plane in the principal stress space that is restricted to the sector \(\sigma_1 \geq \sigma_2 \geq \sigma_3\). Extension, taking into account symmetry conditions, to the other five sectors (see Remark 8.7) defines six planes that constitute a pyramid of indefinite length whose axis is the hydrostatic stress axis (see Figure 8.32). The distance from the origin of the principal stress space to the vertex of the pyramid is \(d = \sqrt{3}c\cot\phi\).

**Remark 8.22.** The particularization \(\phi = 0\) and \(c = \sigma_e/2\) reduces the Mohr-Coulomb criterion to the Tresca criterion (see (8.59) and (8.63)).
Remark 8.23. In soil mechanics, the sign criterion of the normal stresses is the opposite to the one used in continuum mechanics ($\sigma \equiv -\sigma$, see Chapter 4) and, thus, $\sigma_1 \equiv -\sigma_3$ and $\sigma_3 \equiv -\sigma_1$. Then, the Mohr-Coulomb criterion in (8.63) becomes

$$F(\sigma) \equiv (\sigma_1 - \sigma_3) - (\sigma_1 + \sigma_3) \sin \phi - 2c \cos \phi .$$

The corresponding graphical representations are shown in Figures 8.33 and 8.34.

Figure 8.32: Mohr-Coulomb criterion in the principal stress space.

Figure 8.33: Representation of the Mohr-Coulomb criterion using Mohr’s circle in three dimensions and soil mechanics sign criterion.
Remark 8.24. Following certain algebraic operations, the Mohr-Coulomb criterion can be written in terms of the three stress invariants.

Mohr-Coulomb criterion:

\[ F(\sigma) \equiv F(I_1, J_2', J_3') \]

Remark 8.25. The Mohr-Coulomb criterion is especially adequate for cohesive-frictional materials (concrete, rocks and soils), which are known to exhibit considerably different uniaxial elastic limits under tensile and compressive loadings.

8.8.4 Drucker-Prager Criterion

The yield surface defined by the Drucker-Prager criterion is given by

Drucker-Prager criterion:  
\[ F(\sigma) \equiv 3 \alpha \sigma_m + (J_2')^{1/2} - \beta = 0 \]  (8.64)

where
\[ \alpha = \frac{2 \sin \phi}{\sqrt{3} (3 - \sin \phi)} , \quad \beta = \frac{6c \cos \phi}{\sqrt{3} (3 - \sin \phi)} \quad \text{and} \quad \sigma_m = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \frac{I_1}{3} , \quad (8.65) \]

being \( c \) and \( \phi \) the cohesion and the internal friction angle, respectively, which are considered to be material properties. Considering (8.16) and (8.18), the criterion can be rewritten as

\[
F(\sigma) \equiv \alpha I_1 + (J'_2)^{1/2} - \beta = 3 \alpha \sigma_{oct} + \sqrt{\frac{3}{2}} \tau_{oct} - \beta = F(I_1, J'_2) = 0 . \quad (8.66)
\]

**Remark 8.26.** The independence on the third stress invariant \( J'_3 \) establishes that, if a certain point in the stress space belongs to the yield surface, all the other points with the same value of the stress invariants \( I_1 \) and \( J'_2 \) also belong to this surface, independently of the value of the third stress invariant \( J'_3 \). Given that the constant values of these invariants correspond to points of the octahedral plane placed at a same distance from the hydrostatic stress axis (see Figure 8.6), it can be concluded that the yield surface is a surface of revolution around this axis.

In addition, because the relation between \( \sigma_{oct} \) and \( \tau_{oct} \) in (8.66) is lineal, the surface is a conical surface whose axis is the hydrostatic stress axis (see Figure 8.5 and Figure 8.35). The distance from the origin of the principal stress space to the vertex of the cone is \( d = \sqrt{3} c \cot \phi \). It can be verified that the Drucker-Prager surface has the Mohr-Coulomb surface with the same values of cohesion, \( c \), and internal friction angle, \( \phi \), semi-inscribed in it.

Figure 8.35: Drucker-Prager criterion in the principal stress space.
Remark 8.27. The position of the vertex of the Drucker-Prager cone in the positive side of the hydrostatic stress axis establishes a limitation in the elastic behavior range for hydrostatic stress states in tensile loading (while there is no limitation in the elastic limit for the hydrostatic compression case). This situation, which also occurs in the Mohr-Coulomb criterion, is typically observed in cohesive-frictional materials (concrete, rocks and soils), for which these two criteria are especially adequate.

Remark 8.28. In soil mechanics, where the sign criterion for the normal stresses is inverted, the yield surface for the Drucker-Prager criterion is as indicated in Figure 8.36.

Remark 8.29. The particularization $\phi = 0$ and $c = \sigma_e/2$ reduces the Drucker-Prager criterion to the von Mises criterion (see (8.56), (8.64) and (8.65)).

Figure 8.36: Drucker-Prager criterion in the principal stress space, using soil mechanics sign criterion.
**Problem 8.1** – Justify the shape the yield surface will have in the principal stress space for each of the following cases:

a) \( f(I_1^2) = 0 \)

b) \( f(J'_2) = 0 \)

c) \( aI_1^2 + b \tau_{oct}^2 = c \) with \( a, b \) and \( c \) strictly positive

**Solution**

a) In this case, there is a condition on the mean stress since

\[ I_1 = \sigma_1 + \sigma_2 + \sigma_3 = 3\sigma_m. \]

Then, the yield surface is an octahedral plane whose distance to the origin is imposed by the first stress invariant. However, because this invariant is squared, there are two octahedral planes, one in each direction of the hydrostatic stress axis.

\[ \sigma_3 \quad \sqrt{3}\sigma_m \]

\[ \sqrt{3}\sigma_m \quad \sigma_2 \]

hydrostatic stress axis

b) Here, the distance between a given stress state and an hydrostatic stress state is imposed. So, the yield surface is a cylinder with circular section in the octahedral planes,

\[ J'_2 = \frac{3}{2} \tau_{oct}^2 \implies \text{distance} = \sqrt{3}\tau_{oct}. \]
c) The representation of a plane defined by a given point of the yield surface and the hydrostatic stress axis is:

\[
\begin{align*}
P & = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
R & = \begin{pmatrix} \sqrt{3} \sigma_{oct} \\ \sqrt{3} \tau_{oct} \end{pmatrix}
\end{align*}
\]

Then, the relations

\[
\begin{align*}
d &= x = \sqrt{3} \sigma_{oct} = \frac{\sqrt{3}}{3} I_1 \\
R &= y = \sqrt{3} \tau_{oct}
\end{align*}
\]

are deduced and replacing these values in the given expression of the yield surface results in

\[
a I_1^2 + b \tau_{oct}^2 = c \quad \Rightarrow \quad 3ax^2 + \frac{by^2}{3} = c \quad \Rightarrow \quad \left( \frac{x}{\sqrt{3a}} \right)^2 + \left( \frac{y}{\sqrt{3c}} \right)^2 = 1.
\]

This is the mathematical description of an ellipse in the \( x - y \) plane previously defined. In addition, since the third stress invariant does not intervene in the definition of the yield surface, the hydrostatic stress axis is an axis of radial symmetry and, thus, the rotation of the ellipse about the \( x \)-axis (\( \equiv \) hydrostatic stress axis) defines the final surface. In conclusion, if the axes considered are the axes \( x (\equiv \) hydrostatic stress axis), \( y \) and \( z \), the yield surface is defined by
\[
\left( \frac{x}{\sqrt{3a}} \right)^2 + \left( \frac{y}{\sqrt{3c}} \right)^2 + \left( \frac{z}{\sqrt{3b}} \right)^2 = 1.
\]

**Problem 8.2** – Graphically determine, indicating the most significant values, the cohesion and internal friction angle of an elastoplastic material that follows the Mohr-Coulomb yield criterion using the following information:

1) In an uniaxial tensile stress state \((\sigma_1 = \sigma, \sigma_2 = \sigma_3 = 0)\), the material plasticizes at \(\sigma = \sigma_A\).

2) In a triaxial isotensile test of the same material \((\sigma_1 = \sigma_2 = \sigma_3 = \sigma)\), it plasticizes at \(\sigma = \sigma_B\).

**Solution**

In the uniaxial tensile stress state, the Mohr’s circle will cross the origin and the value \(\sigma = \sigma_A\) in the horizontal axis. However, for the triaxial isotensile stress state, the Mohr’s circle will degenerate to a point in this axis, \(\sigma = \sigma_B\). Thus, the following graph is plotted.
which allows establishing the relations
\[ \tan \phi = \frac{c}{\sigma_B} \quad \text{and} \quad \sin \phi = \frac{\sigma_A/2}{\sigma_B - \sigma_A/2}. \]

Finally, the cohesion and internal friction angle are
\[ \phi = \arcsin \frac{\sigma_A/2}{\sigma_B - \sigma_A/2} \quad \text{and} \quad c = \sigma_B \tan \phi. \]

**Problem 8.3** – The following properties of a certain material have been experimentally determined:

1) In a hydrostatic compressive regime, the material never plasticizes.

2) In a hydrostatic tensile regime, the virgin material plasticizes for a value of the mean stress \( \sigma_m = \sigma^\ast \).

3) In an uniaxial tensile regime, the virgin material plasticizes for a tensile stress value \( \sigma_u \).

4) In other cases, plasticization occurs when the norm of the deviatoric stresses varies linearly with the mean stress,
\[ |\sigma'| = \sqrt{\sigma' : \sigma'} = a\sigma_m + b. \]

Plot the yield surface, indicating the most significant values, and calculate the values \( a \) and \( b \) in terms of \( \sigma^\ast \) and \( \sigma_u \).

**Solution**

Property 1) and 2) indicate that the yield surface is closed in the tensile part of the hydrostatic stress axis but open in the compressive part. In addition, property 3) indicates that the octahedral plane that contains the origin will have the shape shown in the figure to the right. Since property 4) indicates that the deviatoric stresses vary linearly with the mean stress (as is the case for the Drucker-Prager criterion), then the yield surface is necessarily a right circular cone whose axis is the hydrostatic...
stress axis and whose vertex is in the tensile part of this axis:

\[ \sigma_3 \]

\[ \sqrt{3} \sigma_m = \sqrt{3} \sigma^* \]

To calculate the values of \( a \) and \( b \), the yield criterion \( |\sigma'| = \sqrt{\sigma' : \sigma'} = a \sigma_m + b \) is applied on the vertex of the cone, which corresponds with the hydrostatic tensile case and, thus, has no deviatoric stresses.

\[ |\sigma'| = 0 \quad \implies \quad a \sigma_m \bigg|_{\sigma_m=\sigma^*} + b = 0 \quad \implies \quad a \sigma^* + b = 0 \tag{1} \]

The procedure is repeated for the uniaxial tensile case, whose deviatoric stresses are now

\[ \sigma_{\text{not}} = \begin{bmatrix} \sigma_u & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \sigma_{\text{sph}} = \frac{1}{3} \sigma_u \mathbf{1} \implies \sigma' = \frac{\sigma_u}{3} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \]

Then, applying the yield criterion \( |\sigma'| = a \sigma_m + b \) produces

\[ |\sigma'| = \sqrt{\frac{2}{3} \sigma_u} \quad \implies \quad \sqrt{\frac{2}{3} \sigma_u} = a \left( \frac{1}{3} \sigma_u \right) + b \tag{2} \]

Equations [1] and [2] allow determining the desired values of \( a \) and \( b \) as

\[ a = \frac{\sqrt{\frac{2}{3} \sigma_u}}{\frac{\sigma_u}{3} - \sigma^*} \quad \text{and} \quad b = -\frac{\sqrt{\frac{2}{3} \sigma_u \sigma^*}}{\frac{\sigma_u}{3} - \sigma^*}. \]
Problem 8.4 – The metallic component PQRS has a thickness “e” and is composed of two different materials, (1) and (2), considered to be perfect elasto-plastic materials. The component is subjected to a pure shear test by means of the machine shown in Figure A, such that the uniform stress and strain states produced are

\[
\varepsilon_x = \varepsilon_y = \varepsilon_z = 0, \quad \gamma_{xz} = \gamma_{yz} = 0, \quad \gamma_{xy} = \gamma = \frac{\delta}{h}, \\
\sigma_x = \sigma_y = \sigma_z = 0, \quad \tau_{xz} = \tau_{yz} = 0 \quad \text{and} \quad \tau_{xy} = \tau \neq 0.
\]

When a component exclusively composed of one of the materials is tested separately, a \( \tau - \gamma \) curve of the type shown in Figure B is obtained for both materials. Determine:

a) The elastic limit that will be obtained in separate uniaxial tensile tests of each material, assuming they follow the von Mises criterion.

When the component composed of the two materials is tested, the \( P - \delta \) curve shown in Figure C is obtained. Determine:

b) The values of the elastic load and displacement, \( P_e \) and \( \delta_e \).

c) The values of the plastic load and displacement, \( P_p \) and \( \delta_p \).

d) The coordinates \( P - \delta \) of points C and D in Figure C.

**HYPOTHESES:**

Material (1)
\[ G = G \quad \text{and} \quad \tau_e = \tau^* \]

Material (2)
\[ G = G \quad \text{and} \quad \tau_e = 2 \tau^* \]
Solution

a) In an uniaxial state of stress, plasticization according to the von Mises criterion is known to begin when (see Example 8.2)

$$ \bar{\sigma} = \sigma_e, $$

where $\bar{\sigma}$ is the effective stress and $\sigma_e$ is the elastic limit. In addition, the following relations seen in this chapter, are known to hold.

$$ \bar{\sigma} = (3J'_2)^{\frac{1}{2}}, \quad J'_2 = \frac{1}{2} \text{Tr}(\sigma')^2, $$

$$ \sigma' = \sigma - \sigma_{\text{sph}}, \quad \sigma_{\text{sph}} = \sigma_m \mathbf{1}, \quad \sigma_m = \frac{1}{3} \text{Tr}(\sigma). $$

For this problem in particular,

$$ \sigma \equiv \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, $$

so $\sigma_m = 0$ and, therefore, $\sigma_{\text{sph}} = 0$, leading to $\sigma' = \sigma$. Then,

$$ (\sigma')^2 \equiv \begin{bmatrix} \tau^2 & 0 & 0 \\ 0 & \tau^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad J'_2 = \tau^2 \implies \bar{\sigma} = \sqrt{3} \tau. $$

Considering that material (1) plasticizes when $\tau_e = \tau^*$ and material (2), when $\tau_e = 2\tau^*$, then

Material 1 $\implies \sigma_e = \sqrt{3} \tau^*$,

Material 2 $\implies \sigma_e = 2\sqrt{3} \tau^*$.

b) The elastic load $P_e$ and the elastic displacement $\delta_e$ determine the end of the elastic regime in the component. The statement of the problem indicates that when the materials are tested separately, the $\tau - \gamma$ curve in Figure B is obtained, where $\tau_e = \tau^*$ in material (1) and $\tau_e = 2\tau^*$ in material (2). It is also known that $G$ is the same in both materials, that is, they have the same slope in their respective $\tau - \gamma$ curves.

Now, to determine the combined behavior of these materials in the metallic component, one can assume that the behavior will be elastic in this component as long as both materials are in their corresponding elastic domain. Therefore,
since the elastic interval of material (1) is smaller, then this material will define the elastic domain of the whole component (up to point A in Figure C).

To obtain the value of the elastic force, equilibrium of forces is imposed for the force $P_e$ and the stresses each material has at point A. Note that equilibrium is imposed on forces, therefore, stresses must be multiplied by the surface on which they act, considering the magnitude perpendicular to the plane of the paper as the unit value.

$$P_e = \frac{h}{2} \tau_{A1} + \frac{h}{2} \tau_{A2} = \frac{h}{2} \tau^* + \frac{h}{2} \tau^* = P_e = h \tau^*$$

The elastic displacement is obtained imposing kinematic compatibility of the two materials,

$$\delta_e = \gamma_1^A h = \gamma_2^A h \implies \delta_e = \frac{\tau^*}{G} h.$$

c) To obtain the plastic values $P_p^1$ and $\delta_p$ one must take into account that, at point A, material (1) begins to plasticize, while material (2) initiates plasticization at point B. Therefore, the behavior of the complete component will be perfectly plastic starting at point B, but elastoplastic between points A and B. To determine the coordinates of point B, the same procedure as before is used. Plotting the $\tau - \gamma$ curves of each separate material up to point B results now in

\[\text{Figure showing \tau - \gamma curves for both materials.}\]
and, imposing the equilibrium and compatibility equations, yields the values of $P_p$ and $\delta_p$.

\[
\begin{align*}
P_p &= \frac{h}{2} \tau B_1 + \frac{h}{2} \tau B_2 = \frac{h}{2} \tau^* + \frac{h}{2} 2\tau^* \\
\delta_p &= \gamma B_1 h = \gamma B_2 h = \frac{2\tau^*}{G} h
\end{align*}
\]

\[\Rightarrow P_p = \frac{3}{2} \tau^* h \quad \text{and} \quad \delta_p = \frac{2\tau^* h}{G} = 2\delta_e.\]

d) The coordinates of points $A$ and $B$ have already been obtained. The statement of the problem gives the value of point $B'$, which corresponds to a deformation of $3\delta_e$ when the plastic load $P_p$ is maintained constant (perfectly plastic regime).

Consider first the material (1). Unloading takes place starting at $B'$ and, according to the information given, this material plasticizes when it reaches a value of $-\tau^*$. The slope of the curve is still the value of the material parameter $G$ since this is independent of the material being under loading or unloading conditions. Thus, to determine point $C$ it is enough to draw a straight line that crosses point $B'$ and is parallel to $OA$, until the value $-\tau^*$ is reached.

The same occurs in the case of material (2), with the difference that when the line parallel to $OA$ is drawn to cross point $B'$, this line must be extended to the value $-2\tau^*$ (which corresponds to point $D$).

Then, the load and displacement values at point $B'$ are

\[
\delta B' = 3\delta_A = \frac{3\tau^* h}{G} = 3\delta_e \quad \text{and} \quad P B' = P_B = \frac{3}{2} \tau^* h.
\]
To obtain the load and displacement values at point C, the equilibrium and compatibility equations are imposed. Taking into account the $\tau$ and $\gamma$ values obtained at point C in the curves above yields

\[ P_C = \frac{h}{2} \tau_C^1 + \frac{h}{2} \tau_C^2 = \frac{h}{2} (-\tau^*) + \frac{h}{2} (0) = -\frac{\tau^* h}{2} \]
\[ \delta_C = \gamma_C^1 h = \left( \frac{\tau^*}{G} \right) h \]

\[ \Rightarrow \begin{cases} P_C = -\frac{\tau^* h}{2} \text{ and } \\ \delta_C = \frac{\tau^* h}{G} = \delta_e. \end{cases} \]

Repeating the procedure for point D results in

\[ P_D = \frac{h}{2} \tau_D^1 + \frac{h}{2} \tau_D^2 = \frac{h}{2} (-\tau^*) + \frac{h}{2} (-2\tau^*) = -\frac{3\tau^* h}{2} \]
\[ \delta_D = \gamma_D^2 h = \left( -\frac{\tau^*}{G} \right) h \]

\[ \Rightarrow \begin{cases} P_D = -\frac{3\tau^* h}{2} \text{ and } \\ \delta_D = -\frac{\tau^* h}{G} = -\delta_e. \end{cases} \]

**Problem 8.5** – Consider the solid cylinder shown in Figure A, which is fully fixed at its base and has a torsional moment $M$ applied on its top end. The cylinder is composed of two materials, (1) and (2), which have an elastoplastic tangent stress-strain behavior, as shown in Figure B. Assume the following displacement field in cylindrical coordinates (Coulomb torque),

\[ u(r, \theta, z) \equiv [u_r, \ u_{\theta}, \ u_z]^T = \left[ 0, \ \frac{\theta}{h} r z, \ 0 \right]^T, \]

where $\phi$ is the rotation of the section at the free end of the cylinder. Assuming infinitesimal strains, determine:

a) The strain and stress tensors, $\varepsilon$ and $\sigma$, in cylindrical coordinates and elastic regime. Plot, indicating the most significant values, the $\sigma_r - r$ and $\tau_{\theta z} - r$ curves for a cross-section of the cylinder at height $z$. Schematically represent the stress distribution of $\tau_{\theta z}$ in this cross-section.

b) The value of $\phi = \phi_e$ (see Figure C) for which plasticization begins in at least one point of the cylinder, indicating where it begins and the corresponding value of the moment $M = M_e$.

**NOTE:** $M = \int_S r \tau_{\theta z} dS$

c) The minimum value of $\phi = \phi_1$ for which material (1) has totally plasticized and the corresponding value of $M = M_1$ (see Figure C). Schematically represent the stress distribution in a cross-section at this instant.
d) The minimum value of $\phi = \phi_2$ for which material (2) has totally plasticized and the corresponding value of $M = M_2$ (see Figure C). Schematically represent the stress distribution in a cross-section at this instant.

e) The asymptotic value of $M = M_p$ (= plastic moment) corresponding to the plasticization of the complete cross-section. Schematically represent the stress distribution in a cross-section at this instant.

![Figure A](image1.png)

![Figure C](image2.png)

**Hypotheses:**
Material (1): $G = G$ and $\tau_e = \tau^*$.  
Material (2): $G = G$ and $\tau_e = 2\tau^*$.

**Solution**

a) The infinitesimal strain tensor is calculated directly from the given displacement field, both in cylindrical coordinates,

$$
\varepsilon = \begin{bmatrix} 0 & 0 & 0 \\
0 & 0 & \frac{\phi r}{2h} \\
0 & \frac{\phi r}{2h} & 0 \end{bmatrix}.
$$
To compute the stress tensor, the constitutive equation of an isotropic elastic material is used. Note that the two materials composing the cylinder have the same parameter $G$, then

$$\sigma = \lambda \text{Tr}(\varepsilon) 1 + 2\mu \varepsilon$$

and

$$\text{Tr}(\varepsilon) = 0 \implies \mu = G \implies \sigma = 2G\varepsilon.$$

The stress tensor results in

$$\sigma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{G\phi_r}{h} \\ 0 & \frac{G\phi_r}{h} & 0 \end{bmatrix}.$$

Plotting the $\sigma_{rr}$ and $\tau_{\theta z}$ components of the stress tensor in terms of the radius $r$ yields:

The stresses are linear and do not depend on the $z$-coordinate of the cross-section considered. Thus, the distribution of stresses in any cross-section ($z = \text{const}$.) of the cylinder is:

$$0 \leq \phi \leq \phi_r.$$
b) Given the stress distribution \( \tau = (G\phi r/h) \leq \phi \leq \phi_e \), the moment acting on the cylinder is

\[
M = \int_S r \tau(r) dS = \int_0^R \int_0^R \left(\frac{G\phi r}{h}\right) r dr d\theta = 2\pi \frac{G\phi R^3}{h} dr = \frac{\pi GR^4}{2h} \phi. \quad [1]
\]

This is the relation between the moment and the rotation angle \( (M - \phi) \) at the free end of the cylinder when the two materials behave elastically.

Material (1) starts to plasticize first at \( \tau_e = \tau^* \), since material (2) plasticizes at a higher stress, \( \tau_e = 2\tau^* \). In addition, the external surface of the cylinder \( (r = R) \) suffers larger stresses, and this surface is composed of material (1). Therefore, plasticization will initiate when

\[
\tau \Big|_{r=R; \phi=\phi_e} = \tau^* \quad \implies \quad \frac{G\phi_e R}{h} = \tau^* \quad \implies \quad \phi_e = \frac{\tau^* h}{GR}.
\]

is satisfied. This is the value of the rotation angle at the free end of the cylinder required for plasticization to initiate in the exterior material points of the cylinder (material (1)). The corresponding moment is obtained by replacing \( \phi_e \) in [1],

\[
M_e = M(\phi_e) = \frac{\pi GR^4}{2h} \phi_e = \frac{\pi \tau^* R^3}{2}.
\]

c) If the material were elastic, the slope of the stresses \( \tau \) would increase with \( \phi \) (remaining, though, linear with \( r \)), but since the material is now elastoplastic, stresses cannot exceed the value \( \tau_e \), which corresponds to the onset of plasticity. Then, the limit value is obtained for \( \tau_e = \tau^* \) when \( \phi = \phi_1 \) for \( (R/2) \leq r \leq R \). That is, material (1) has a perfectly plastic distribution of stresses while material (2) remains elastic.
The following condition is imposed to compute this rotation $\phi_1$.

\[
\tau \bigg|_{r=R/2; \phi=\phi_1} = \tau^* \quad \implies \quad \frac{G\phi_1 R}{2h} = \tau^* \quad \implies \quad \phi_1 = \frac{2\tau^* h}{GR}.
\]

This is the minimum value of the rotation angle at the free end of the cylinder required for material (1) to be completely plasticized.

In order to compute the corresponding moment, relation [1] between $M$ and $\phi$ is no longer valid here because material (1) behaves elastoplasticly while material (2) behaves completely elastically. The moment acting on the cylinder is now

\[
M_1 = 2\pi \int_0^{R/2} r \tau^* r dr d\theta + \int_0^{R/2} r \left( \frac{G\phi_1 r}{h} \right) r dr d\theta = 2\pi \tau^* \int_{R/2}^{R} r^2 dr + 2\pi G \frac{\phi_1}{h} \int_0^{R/2} r^3 dr = \frac{31}{48} \pi \tau^* R^3.
\]

\[d)\] Material (2) starts plasticizing for $\tau_e = 2\tau^*$, which does not correspond with the end of plasticization in material (1) at $\tau_e = \tau^*$. Then, the stress distribution for $\phi = \phi_2$ (onset of plasticization in material (2)) is

The following condition is imposed to obtain the value of the rotation angle.

\[
\tau \bigg|_{r=R/2; \phi=\phi_2} = 2\tau^* \quad \implies \quad \frac{G\phi_2 R}{2h} = 2\tau^* \quad \implies \quad \phi_2 = \frac{4\tau^* h}{GR}.
\]
The corresponding moment is

\[ M_2 = \int_0^{2\pi} \int_{R/2}^{R} r r^* \, r \, d\theta \, dr + \int_0^{2\pi/2} \int_0^{R/2} \left( \frac{G\phi_2 r}{h} \right) r \, dr \, d\theta \implies M_2 = \frac{17}{24} \pi \tau^* R^3. \]

e) The asymptotic value of \( M(M_p) \) corresponds to the total plasticization of the cylinder. The stress distribution in this case is:

Through integration, the corresponding moment is obtained,

\[ M_p = \int_0^{2\pi} \int_{R/2}^{R} r r^* \, r \, d\theta \, dr + \int_0^{2\pi/2} \int_0^{R/2} r 2\tau^* \, r \, dr \, d\theta \implies M_p = \frac{3}{4} \pi \tau^* R^3. \]
**Exercises**

8.1 – Formulate in terms of the stress invariants $I_1$, $J_2'$ and $J_3'$ the equation of the yield surface that, in the principal stress space, is a spheroid (ellipsoid of revolution) with semi-axes $a$ and $b$.

![Diagram of spheroid with semi-axes $a$ and $b$]

Intersection with octahedral plane at $(0,0,0)$

8.2 – An elastoplastic material is subjected to a pure shear test (I) and an uniaxial tensile test (II). Plasticization occurs, respectively, at $\tau = a$ and $\sigma = b$. Determine the values of the cohesion and internal friction angle assuming a Mohr-Coulomb yield criterion.

![Diagram of pure shear (I) and uniaxial tension (II)]

8.3 – A component ABCD of a perfectly elastoplastic material is tested in the machine illustrated in Figure A. The action-response curve $(P - \delta)$ obtained is shown in Figure B. An uniaxial stress-strain state is assumed such that

\[
\varepsilon_x = \frac{\delta}{hL_y}, \quad \varepsilon_y = \varepsilon_z = \gamma_{xy} = \gamma_{xz} = \gamma_{yz} = 0, \\
\sigma_x \neq 0 \quad \text{and} \quad \sigma_y = \sigma_z = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0.
\]
Determine the following values, indicated in the curve of Figure B:

a) The elastic load $P_e$ and the corresponding displacement $\delta_e$.

b) The ultimate plastic loads for tensile and compressive loadings, $P_p$ and $P_q$, respectively.

c) The values of $P$ and $\delta$ at points (1) and (2).

Additional hypotheses:

1) Young’s modulus, $E$, and Poisson’s coefficient, $\nu$.

2) Elastic limit, $\sigma_e$.

3) Thickness of the component, $b$. 
8.4 – The truss structure OA, OB and OC is composed of concrete, which is assumed to behave as a perfectly elastoplastic material with a tensile elastic limit $\sigma_e$ and a compressive elastic limit $10\sigma_e$. An increasing vertical load $P$ is applied at point O, starting at $P = 0$, until a vertical displacement $\delta = 20\sigma_e L/E$ is reached at this point. Then, the load is decreased back to $P = 0$.

a) Draw the $P - \delta$ diagram of the process, indicating the most significant values and the state of plasticization of the bars at each instant.

b) Calculate the displacement value at point O at the end of the process.

8.5 – Consider a solid sphere with radius $R_1$ encased inside a spherical shell with interior radius $R_1$ and exterior radius $R_2$. The sphere and the shell are composed of the same material and are initially in contact without exerting any pressure on each other. At a certain moment, the interior sphere is heated up to a temperature increment $\Delta\theta$.

Determine:

a) The value of the exterior pressure required on the shell for said shell to keep a constant value (infinitesimal strain hypothesis).

b) The displacement, strain and stress fields in both the sphere and the shell under these conditions.

c) The minimum value of $\Delta\theta$ for which plasticization initiates in some point, assuming the aforementioned conditions and considering a von Mises criterion.
Additional hypotheses:

1) Material properties:
   - Young’s modulus, $E$, and Poisson’s coefficient, $\nu = 0$.
   - Thermal constant, $\alpha$.
   - Yield stress, $\sigma_y$.
   - Radii, $R_1 = 1$ and $R_2 = 3$.

2) The body forces are negligible.

3) The displacement and stress fields of a spherical shell with interior radius $R_i$ and exterior radius $R_e$ subjected to an interior pressure $P_i$ and an exterior pressure $P_e$ are, for $\nu = 0$:

$$ u = \begin{bmatrix} u_r(r) \\ 0 \\ 0 \end{bmatrix} \quad u_r = C r + \frac{C_1}{r^2}; \quad C = \frac{P_i R_i^3 - P_e R_e^3}{E (R_e^2 - R_i^2)}; \quad C_1 = \frac{P_i - P_e}{2E} \frac{R_i^3 R_e^3}{R_e^3 - R_i^3}$$

$$ \sigma = \begin{bmatrix} \sigma_{rr} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{\phi\phi} \end{bmatrix} \quad \sigma_{rr} = E \left( C - \frac{2C_1}{r^2} \right); \quad \sigma_{\theta\theta} = \sigma_{\phi\phi} = E \left( C + \frac{C_1}{r^2} \right)$$

8.6 – Consider a solid sphere with radius $R_1$ and composed of material (1), encased inside a spherical shell with interior radius $R_1$, exterior radius $R_2$ and composed of material (2). The sphere and the shell are initially in contact without exerting any pressure on each other. An exterior pressure $P$ is applied simultaneously with a temperature increment $\Delta \theta$.

a) Determine the possible values of $\Delta \theta$ and $P$ (positive or negative) for which the contact (without exerting any pressure) between the sphere and the shell is maintained. Plot the corresponding $P - \Delta \theta$ curve.

b) Obtain the stress state of the shell and the sphere for these values.

c) Under these conditions, compute, for each value of the pressure $P$, the value of $\Delta \theta^*$ for which plasticization initiates at some point of the sphere or the shell, according to the von Mises and Mohr-Coulomb criteria. Plot the corresponding $P - \Delta \theta^*$ curves (interaction graphs).

Additional hypotheses:

1) Material properties:
   - Young’s moduli, $E^{(1)} = E^{(2)} = E$, and Poisson’s coefficients, $\nu^{(1)} = \nu^{(2)} = 0$.
   - Thermal constants, $\alpha^{(1)} = 2\alpha$ and $\alpha^{(2)} = \alpha$.
   - Yield stresses, $\sigma^{(1)}_y = \sigma^{(2)}_y = \sigma_y$. 

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− Cohesion values, $C^{(1)} = C^{(2)} = C$, and internal friction angles, $\phi^{(1)} = \phi^{(2)} = 30^\circ$.
− Radii, $R_1 = 1$ and $R_2 = 2$.

2) The displacement and stress fields of a spherical shell with interior radius $R_i$ and exterior radius $R_e$ subjected to an interior pressure $P_i$ and an exterior pressure $P_e$ are, for $\nu = 0$:

\[
\mathbf{u} = \begin{bmatrix} u_r \left( r \right) \\ 0 \\ 0 \end{bmatrix} \quad u_r = Cr + \frac{C_1}{r^2}; \quad C = \frac{P_i R_i^3 - P_e R_e^3}{E \left( R_e^3 - R_i^3 \right)}; \quad C_1 = \frac{P_i - P_e}{2E} \frac{R_i^3 R_e^3}{R_e^3 - R_i^3}
\]

\[
\mathbf{\sigma} = \begin{bmatrix} \sigma_{rr} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{\phi\phi} \end{bmatrix}
\]

\[
\sigma_{rr} = E \left( C - \frac{2C_1}{r^3} \right); \quad \sigma_{\theta\theta} = \sigma_{\phi\phi} = E \left( C + \frac{C_1}{r^3} \right)
\]

8.7 – A cylinder of radius $R$ and height $h$ is subjected to an exterior load $P$ and a uniform temperature increment $\Delta \theta$.

a) Determine the displacement, strain and tensor fields in terms of the integration constants.
b) Determine the integration constants and the corresponding displacement, strain and tensor fields.
c) Given $p = p^* > 0$, determine the corresponding value of $\Delta \theta^*$ such that there are no horizontal displacements.
d) Under the conditions described in c), determine the value of $p^*$ for which the cylinder begins to plasticize according to the Mohr-Coulomb criterion.

Additional hypotheses:

1) Material properties:
   − Cohesion value, $C$, and internal friction angle, $\phi = 30^\circ$.
   − Thermal constant, $\beta$.
   − Lamé parameters, $\lambda = \mu$.
2) The body forces are negligible.
3) The friction between the cylinder and the ground can be neglected.