CH.7. PLANE LINEAR ELASTICITY

Multimedia Course on Continuum Mechanics
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7.1 Plane Linear Elasticity Theory

Ch.7. Plane Linear Elasticity
For some problems, one of the principal directions is known \textit{a priori}:

- Due to particular geometries, loading and boundary conditions involved.
- The elastic problem can be solved independently for this direction.
- Setting the known direction as \( z \), the elastic problem analysis is reduced to the \( x-y \) plane \( \Rightarrow \) \textbf{PLANE ELASTICITY}

There are two main classes of plane linear elastic problems:

- Plane stress
- Plane strain

\textbf{REMARK}

The isothermal case will not be studied here for the sake of simplicity. Generalization of the results obtained to thermo-elasticity is straight-forward.
7.2 Plane Stress

Ch.7. Plane Linear Elasticity
Simplifying hypothesis of a plane stress linear elastic problem:

1. Only stresses “contained in the $x$-$y$ plane” are not null.

\[
\begin{bmatrix}
\sigma_x & \tau_{xy} & 0 \\
\tau_{xy} & \sigma_y & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

2. The stress are independent of the $z$ direction.

\[
\begin{aligned}
\sigma_x &= \sigma_x (x, y, t) \\
\sigma_y &= \sigma_y (x, y, t) \\
\tau_{xy} &= \tau_{xy} (x, y, t)
\end{aligned}
\]

**REMARK**

The name “plane stress” arises from the fact that all (not null) stress are contained in the $x$-$y$ plane.
These hypothesis are valid when:

- The thickness is much smaller than the typical dimension associated to the plane of analysis: \( e << L \)

- The actions \( b(x,t), u^*(x,t) \) and \( t^*(x,t) \) are contained in the plane of analysis (in-plane actions) and independent of the third dimension, \( z \).

- \( t^*(x,t) \) is only non-zero on the contour of the body’s thickness:
Strain Field in Plane Stress

- The strain field is obtained from the inverse Hooke’s Law:

\[
\varepsilon = -\frac{\nu}{E} \text{Tr}(\sigma) \mathbf{1} + \frac{1+\nu}{E} \sigma
\]

with

\[
\sigma_z = 0 \quad \tau_{xz} = 0 \quad \tau_{yz} = 0
\]

- As

\[
\begin{align*}
\sigma_x &= \sigma_x(x, y, t) \\
\sigma_y &= \sigma_y(x, y, t)
\end{align*}
\]

- And the **strain tensor** for plane stress is:

\[
\varepsilon(x, y, t) \equiv \begin{bmatrix}
\varepsilon_x & \frac{1}{2} \gamma_{xy} & 0 \\
\frac{1}{2} \gamma_{xy} & \varepsilon_y & 0 \\
0 & 0 & \varepsilon_z
\end{bmatrix}
\]

with

\[
\varepsilon_z = -\frac{\nu}{1-\nu} \left( \varepsilon_x + \varepsilon_y \right)
\]
Constitutive equation in Plane Stress

Operating on the result yields:

\[
\begin{align*}
\varepsilon_x &= \frac{1}{E} (\sigma_x - \nu \sigma_y) \\
\varepsilon_y &= \frac{1}{E} (\sigma_y - \nu \sigma_x) \\
\varepsilon_z &= -\frac{\nu}{E} (\sigma_x + \sigma_y)
\end{align*}
\]

\[
\begin{align*}
\gamma_{xy} &= 2 \varepsilon_{xy} = \frac{2(1 + \nu)}{E} \tau_{xy} \\
\gamma_{xz} &= 2 \varepsilon_{xz} = 0 \\
\gamma_{yz} &= 2 \varepsilon_{yz} = 0
\end{align*}
\]

\[
\begin{align*}
\sigma_x &= \frac{E}{(1 - \nu^2)} \left[ \varepsilon_x + \nu \varepsilon_y \right] \\
\sigma_y &= \frac{E}{(1 - \nu^2)} \left[ \varepsilon_y + \nu \varepsilon_x \right] \\
\tau_{xy} &= \frac{E}{2(1 + \nu)} \gamma_{xz}
\end{align*}
\]

Constitutive equation in plane stress (Voigt’s notation)

\[
\{\sigma\} = C^{plane \text{ stress}} \cdot \{\varepsilon\}
\]
Displacement Field in Plane Stress

- The displacement field is obtained from the geometric equations,
  \[ \varepsilon(x,t) = \nabla^S u(x,t) \]. These are split into:

  - Those which do not affect the displacement \( u_z \):
    \[
    \begin{align*}
    \varepsilon_x(x,y,t) &= \frac{\partial u_x}{\partial x} \\
    \varepsilon_y(x,y,t) &= \frac{\partial u_y}{\partial y} \\
    \gamma_{xy}(x,y,t) &= 2\varepsilon_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}
    \end{align*}
    \]

  - Those in which \( u_z \) appears:
    \[
    \begin{align*}
    \varepsilon_z(x,y,t) &= -\frac{\nu}{1-\nu}(\varepsilon_x + \varepsilon_y)(x,y,t) = \frac{\partial u_z}{\partial z} \\
    \gamma_{xz}(x,y,t) &= 2\varepsilon_{xz} = \frac{\partial u_x(x,y)}{\partial z} + \frac{\partial u_z}{\partial x} = 0 \\
    \gamma_{yz}(x,y,t) &= 2\varepsilon_{yz} = \frac{\partial u_y(x,y)}{\partial z} + \frac{\partial u_z}{\partial y} = 0
    \end{align*}
    \]

  \[ \Rightarrow u_z(x,y,z,t) \]

  \[ \Rightarrow u_z(z,t) \]
The problem can be reduced to the two dimensions of the plane of analysis.

- The unknowns are:
  
  \[ \mathbf{u}(x, y, t) = \begin{pmatrix} u_x \\ u_y \end{pmatrix}, \quad \{\mathbf{\varepsilon}\}(x, y, t) = \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}, \quad \{\mathbf{\sigma}\}(x, y, t) = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} \]

- The additional unknowns (with respect to the general problem) are either null, or independently obtained, or irrelevant:

  \[ \sigma_z = \tau_{xz} = \tau_{yz} = \varepsilon_{xz} = \varepsilon_{yz} = 0 \]

  \[ \varepsilon_z = -\frac{\nu}{1-\nu}(\varepsilon_x + \varepsilon_y) \]

  \[ u_z(x, y, z, t) \rightarrow \text{does not appear in the problem} \]

**REMARK**

This is an **ideal** elastic problem because it cannot be exactly reproduced as a particular case of the 3D elastic problem. There is no guarantee that the solution to \( u_x(x, y, t) \) and \( u_y(x, y, t) \) will allow obtaining the solution to \( u_z(x, y, z, t) \) for the additional geometric eqns.
3D problems which are typically assimilated to a plane stress state are characterized by:

- One of the body’s dimensions is significantly smaller than the other two.
- The actions are contained in the plane formed by the two “large” dimensions.
7.3 Plane Strain

Ch.7. Plane Linear Elasticity
Hypothesis on the Displacement Field

- Simplifying hypothesis of a plane strain linear elastic problem:
  1. The displacement field is

\[ \mathbf{u} = \begin{pmatrix} u_x \\ u_y \\ 0 \end{pmatrix} \]

  2. The displacement variables associated to the \( x-y \) plane are independent of the \( z \) direction.

\[
\begin{align*}
  u_x &= u_x(x, y, t) \\
  u_y &= u_y(x, y, t)
\end{align*}
\]
These hypothesis are valid when:

- The body being studied is generated by moving the plane of analysis along a generational line.
- The actions $b(x,t)$, $u^*(x,t)$ and $t^*(x,t)$ are contained in the plane of analysis and independent of the third dimension, $z$.

In the central section, considered as the “analysis section” the following holds (approximately) true:

$$
\begin{align*}
  u_z &= 0 \\
  \frac{\partial u_x}{\partial z} &= 0 \\
  \frac{\partial u_y}{\partial z} &= 0 
\end{align*}
$$
Strain Field in Plane Strain

- The strain field is obtained from the geometric equations:

\[
\begin{align*}
\varepsilon_x(x, y, t) &= \frac{\partial u_x(x, y, t)}{\partial x} \\
\varepsilon_y(x, y, t) &= \frac{\partial u_y(x, y, t)}{\partial y} \\
\gamma_{xy}(x, y, t) &= \frac{\partial u_x(x, y, t)}{\partial y} + \frac{\partial u_y(x, y, t)}{\partial x}
\end{align*}
\]

- And the strain tensor for plane strain is:

\[
\mathbf{\varepsilon}(x, y, t) = \begin{bmatrix}
\varepsilon_x & \frac{1}{2} \gamma_{xy} & 0 \\
\frac{1}{2} \gamma_{xy} & \varepsilon_y & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{align*}
u_z &= 0 \\
\frac{\partial u_x}{\partial z} &= 0 \\
\frac{\partial u_y}{\partial z} &= 0 \\
\gamma_{xz} &= \frac{\partial u_x(x, y, t)}{\partial z} + \frac{\partial u_z}{\partial x} = 0 \\
\gamma_{yz} &= \frac{\partial u_y(x, y, t)}{\partial z} + \frac{\partial u_z}{\partial y} = 0
\end{align*}
\]

**REMARK**

The name “plane strain” arises from the fact that all strain is contained in the x-y plane.
Introducing the strain tensor into Hooke’s Law \( \sigma = \lambda \text{Tr}(\varepsilon) \mathbf{1} + 2G\varepsilon \) and operating on the result yields:

\[
\begin{align*}
\sigma_x &= \lambda (\varepsilon_x + \varepsilon_y) + 2G\varepsilon_x \\
\sigma_y &= \lambda (\varepsilon_x + \varepsilon_y) + 2G\varepsilon_y \\
\sigma_z &= \lambda (\varepsilon_x + \varepsilon_y) = \nu(\sigma_x + \sigma_y)
\end{align*}
\]

\[
\begin{align*}
\tau_{xy} &= G\gamma_{xy} \\
\tau_{xz} &= G\gamma_{xz} = 0 \\
\tau_{yz} &= G\gamma_{yz} = 0
\end{align*}
\]

As

\[
\begin{align*}
\varepsilon_x &= \varepsilon_x(x,y,t) \\
\varepsilon_y &= \varepsilon_y(x,y,t) \\
\varepsilon_z &= \varepsilon_z(x,y,t) \\
\gamma_{xy} &= \gamma_{xy}(x,y,t)
\end{align*}
\]

And the stress tensor for plane strain is:

\[
\sigma(x,y,t) = \begin{bmatrix}
\sigma_x & \tau_{xy} & 0 \\
\tau_{xy} & \sigma_y & 0 \\
0 & 0 & \sigma_z
\end{bmatrix}
\]

with \( \sigma_z = \nu(\sigma_x + \sigma_y) \)
Introducing the values of the strain tensor into the constitutive equation and operating on the result yields:

\[
\sigma = \lambda \text{Tr}(\varepsilon) \mathbf{1} + 2\mu \varepsilon
\]

\[
\sigma = (\lambda + 2G)\varepsilon_x + \lambda\varepsilon_y = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[\varepsilon_x + \frac{\nu}{1-\nu}\varepsilon_y\right]
\]

\[
\sigma_y = (\lambda + 2G)\varepsilon_y + \lambda\varepsilon_x = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[\varepsilon_y + \frac{\nu}{1-\nu}\varepsilon_x\right]
\]

\[
\tau_{xy} = G\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}
\]

Constitutive equation in plane strain (Voigt’s notation)
The problem can be reduced to the two dimensions of the plane of analysis.

- The unknowns are:

\[
\mathbf{u}(x, y, t) = \begin{bmatrix} u_x \\ u_y \end{bmatrix}, \quad \{\mathbf{\varepsilon}\}(x, y, t) = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}, \quad \{\mathbf{\sigma}\}(x, y, t) = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}
\]

- The additional unknowns (with respect to the general problem) are either null or obtained from the unknowns of the problem:

\[
u_z = 0, \\
\varepsilon_z = \gamma_{xz} = \gamma_{yz} = \tau_{xz} = \tau_{yz} = 0, \\
\sigma_z = \nu(\sigma_x + \sigma_y)
\]
Examples of Plane Strain Analysis

- 3D problems which are typically assimilated to a plane strain state are characterized by:
  - The body is generated by translating a generational section with actions contained in its plane along a line perpendicular to this plane.
  - The plane strain hypothesis \( \varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0 \) must be justifiable. This typically occurs when:
    1. One of the body’s dimensions is significantly larger than the other two.
    Any section not close to the extremes can be considered a symmetry plane and satisfies:
    \[
    \begin{align*}
    u_z &= 0 \\
    \frac{\partial u_x}{\partial z} &= 0 \\
    \frac{\partial u_y}{\partial z} &= 0
    \end{align*}
    \]
    \[
    \mathbf{u} = \begin{bmatrix} u_x \\ u_y \\ 0 \end{bmatrix}
    \]
    2. The displacement in \( z \) is blocked at the extreme sections.
3D problems which are typically assimilated to a plane strain state are:

- Pressure pipe
- Tunnel
- Continuous brake shoe
- Solid with blocked $z$ displacements at the ends
7.4 The Plane Linear Elastic Problem

Ch. 7. Plane Linear Elasticity
A lineal elastic solid is subjected to body forces and prescribed traction and displacement.

The Plane Linear Elastic problem is the set of equations that allow obtaining the evolution through time of the corresponding displacements $\mathbf{u}(x,y,t)$, strains $\mathbf{\varepsilon}(x,y,t)$ and stresses $\mathbf{\sigma}(x,y,t)$.

Actions:

On $\Gamma_\sigma$: $\mathbf{t}^* = \begin{cases} t_x^*(x,y,t) \\ t_y^*(x,y,t) \end{cases}$

On $\Gamma_u$: $\mathbf{u}^* = \begin{cases} u_x^*(x,y,t) \\ u_y^*(x,y,t) \end{cases}$

On $\Omega$: $\mathbf{b} = \begin{cases} b_x(x,y,t) \\ b_y(x,y,t) \end{cases}$
The **Plane Linear Elastic Problem** is governed by the equations:

1. **Cauchy’s Equation of Motion.**
   Linear Momentum Balance Equation.

\[
\nabla \cdot \sigma(x,t) + \rho_0 b(x,t) = \rho_0 \frac{\partial^2 u(x,t)}{\partial t^2}
\]
The **Plane Linear Elastic Problem** is governed by the equations:

2. **Constitutive Equation (Voigt’s notation).**
   **Isotropic Linear Elastic Constitutive Equation.**

\[
\sigma(x, t) = C : \varepsilon
\]

2D

\[
\{\sigma\} = C \cdot \{\varepsilon\}
\]

With

\[
\{\sigma\} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}, \quad \{\varepsilon\} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}
\]

and

\[
C = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}
\]

**PLANE STRESS**

\[
\begin{align*}
\bar{E} &= E \\
\bar{\nu} &= \nu
\end{align*}
\]

**PLANE STRAIN**

\[
\begin{align*}
\bar{\nu} &= \nu \\
\bar{\nu} &= \frac{\nu}{(1-\nu)}
\end{align*}
\]
The Plane Linear Elastic Problem is governed by the equations:


\[ \varepsilon(x,t) = \nabla^s u(x,t) = \frac{1}{2} (u \otimes \nabla + \nabla \otimes u) \]

This is a PDE system of 8 eqns -8 unknowns:

- \( u(x,t) \) 2 unknowns
- \( \varepsilon(x,t) \) 3 unknowns
- \( \sigma(x,t) \) 3 unknowns

Which must be solved in the \( \mathbb{R}^2 \times \mathbb{R}_+ \) space.
Boundary Conditions

- **Boundary conditions in space**
  - Affect the spatial arguments of the unknowns
  - Are applied on the contour $\Gamma$ of the solid, which is divided into:
    - Prescribed displacements on $\Gamma_u$:
      \[
      \mathbf{u}^* = \begin{cases} 
        u_x^* = u_x^*(x, y, t) \\
        u_y^* = u_y^*(x, y, t) 
      \end{cases}
      \]
    - Prescribed stresses on $\Gamma_\sigma$:
      \[
      \mathbf{t}^* = \begin{cases} 
        t_x^* = t_x^*(x, y, t) \\
        t_y^* = t_y^*(x, y, t) 
      \end{cases}
      \]
      \[
      \mathbf{t}^* = \mathbf{\sigma} \cdot \mathbf{n} \quad \text{with} \quad \mathbf{n} = \begin{bmatrix} n_x \\ n_y \end{bmatrix} 
      \]
      \[
      \mathbf{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}
      \]
Boundary Conditions

- **INITIAL CONDITIONS** (boundary conditions in time)
  - Affect the time argument of the unknowns.
  - Generally, they are the known values at $t = 0$:
    - Initial displacements:
      $$ \mathbf{u}(x, y, 0) = \begin{bmatrix} u_x \\ u_y \end{bmatrix} = 0 $$
    - Initial velocity:
      $$ \mathbf{\dot{u}}(x, y, 0) = \begin{bmatrix} \dot{u}_x \\ \dot{u}_y \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \mathbf{v}_0(x, y) $$
Ununknowns

- The 8 unknowns to be solved in the problem are:

\[ \mathbf{u}(x, y, t) = \begin{cases} u_x \\ u_y \end{cases}, \quad \mathbf{\varepsilon}(x, y, t) = \begin{bmatrix} \varepsilon_x & \frac{1}{2} \gamma_{xy} \\ \frac{1}{2} \gamma_{xy} & \varepsilon_y \end{bmatrix}, \quad \mathbf{\sigma}(x, y, t) = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \]

- Once these are obtained, the following are calculated explicitly:

**PLANE STRESS**

\[ \varepsilon_z = \frac{\nu}{1-\nu} (\varepsilon_x + \varepsilon_y) \]

**PLANE STRAIN**

\[ \sigma_z = \nu (\sigma_x + \sigma_y) \]
7.5 Representative Curves

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Traditionally, plane stress states were graphically represented with the aid of the following contour lines:

- Isostatics or stress trajectories
- Isoclines
- Isobars
- Maximum shear lines
- Others: isochromatics, isopatches, etc.
Isostatics or Stress Trajectories

- System of curves which are tangent to the principal axes of stress at each material point.
  - They are the envelopes of the principal stress vector fields.
  - There will exist two (orthogonal) families of curves at each point:
    - Isostatics $\sigma_1$, tangents to the largest principal stress.
    - Isostatics $\sigma_2$, tangents to the smallest principal stress.

REMARK
The principal stresses are orthogonal to each other, therefore, so will the two families of isostatics orthogonal to each other.
Singular and Neutral Points

- **Singular point**: characterized by the stress state
  \[
  \begin{align*}
  \sigma_x &= \sigma_y \\
  \tau_{xy} &= 0
  \end{align*}
  \]

- **Neutral point**: characterized by the stress state
  \[
  \sigma_x = \sigma_y = \tau_{xy} = 0
  \]

**REMARK**
In a singular point, all directions are principal directions. Thus, in singular points isostatics tend to lose their regularity and can abruptly change direction.
Consider the general equation of an isostatic curve: \( y = f(x) \)

\[ \tan(2\alpha) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2\tan\alpha}{1 - \tan^2\alpha} \]

\[ \tan\alpha = \frac{dy}{dx} = y' \]

\[ \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2y'}{1 - (y')^2} \]

Solving the 2nd order eq.:

\[ (y')^2 + \frac{\sigma_x - \sigma_y}{\tau_{xy}} y' - 1 = 0 \]

Differential equation of the isostatics

Known this function, the eq. can be integrated to obtain a family of curves of the type:

\[ y = f(x) + C \]
Isoclines

- Locus of the points along which the principal stresses are in the same direction.
  - The principal stress vectors in all points of an isocline are parallel to each other, forming a constant angle $\theta$ with the $x$-axis.
  - These curves can be directly found using photoelasticity methods.
To obtain the general equation of an isocline with angle $\theta$, the principal stress $\sigma_1$ must form an angle $\alpha = \theta$ with the $x$-axis:

$$\text{Equation of the Isoclines}$$

$$\sigma_1 \text{ must form an angle } \alpha = \theta \text{ with the } x\text{-axis:}$$

$$\varphi(x, y)$$

For each value of $\varphi$, the equation of the family of isoclines parameterized in function of $\theta$ is obtained:

$$y = f(x, \theta)$$

REMARK

Once the family of isoclines is known, the principal stress directions in any point of the medium can be obtained and, thus, the isostatics calculated.
Envelopes of the maximum shear stress (in modulus) vector fields.
- They are the curves on which the shear stress modulus is a maximum.
- Two planes of maximum shear stress correspond to each material point, \( \tau_{\text{max}} \) and \( \tau_{\text{min}} \).
  - These planes are easily determined using Mohr’s Circle.

**REMARK**
The two planes form a 45° angle with the principal stress directions and, thus, are orthogonal to each other. They form an angle of 45° with the isostatics.
Consider the general equation of a slip line \( y = f(x) \), the relation

\[
\tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad \text{and} \quad \beta = \alpha + \frac{\pi}{4}
\]

\[
\tan(2\beta) = \tan\left(2\alpha - \frac{\pi}{2}\right) = -\frac{1}{\tan 2\alpha}
\]

Then,

\[
\begin{align*}
\tan(2\beta) &= -\frac{1}{\tan(2\alpha)} = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} = \frac{2\tan\beta}{1 - \tan^2\beta} \\
\tan(\beta) &= \frac{dy}{dx} = \frac{\text{not}}{y'}
\end{align*}
\]

\[
-\frac{\sigma_x - \sigma_y}{2\tau_{xy}} = \frac{2y'}{1 - (y')^2}
\]

\[
\left(y'\right)^2 - \frac{4\tau_{xy}}{\sigma_x - \sigma_y} y' - 1 = 0
\]
Equation of the maximum shear lines

- Solving the 2nd order eq.:

\[ \phi(x,y) \quad \text{Known this function, the eq. can be integrated to obtain a family of curves of the type:} \]

\[ y = f(x) + C \]

Differential equation of the slip lines:

\[ y' = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \pm \sqrt{\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)^2 + 1} \]
Chapter 7
Plane Linear Elasticity

7.1 Introduction

As seen in Chapter 6, from a mathematical point of view, the elastic problem consists in a system of PDEs that must be solved in the three dimensions of space and in the dimension associated with time \( \mathbb{R}^3 \times \mathbb{R}^+ \). However, in certain situations, the problem can be simplified so that it is reduced to two dimensions in space in addition to, obviously, the temporal dimension \( \mathbb{R}^2 \times \mathbb{R}^+ \). This simplification is possible because, in certain cases, the geometry and boundary conditions of the problem allow identifying an irrelevant direction (associated with a direction of the problem) such that solutions independent of this dimension can be posed \textit{a priori} for this elastic problem.

Consider a local coordinate system \( \{x, y, z\} \) in which the aforementioned irrelevant direction (assumed constant) coincides with the \( z \)-direction. Then, the analysis is reduced to the \( x-y \) plane and, hence, the name \textit{plane elasticity} used to denote such problems. In turn, these are typically divided into two large groups associated with two families of simplifying hypotheses, \textit{plane stress} problems and \textit{plane strain} problems.

For the sake of simplicity, the \textit{isothermal} case will be considered here, even though there is no intrinsic limitation to generalizing the results that will be obtained to the thermoelastic case.

7.2 Plane Stress State

The plane stress state is characterized by the following simplifying hypotheses:

\begin{enumerate}
    \item \textit{The stress state is of the type}
\begin{equation}
    \begin{bmatrix}
    \sigma_x & \tau_{xy} & 0 \\
    \tau_{yx} & \sigma_y & 0 \\
    0 & 0 & 0
    \end{bmatrix}
    \end{equation}
\end{enumerate}
2) The non-zero stresses (that is, those associated with the x-y plane) do not depend on the z-variable,

\[ \sigma_x = \sigma_x(x, y, t), \quad \sigma_y = \sigma_y(x, y, t) \quad \text{and} \quad \tau_{xy} = \tau_{xy}(x, y, t). \]  \hfill (7.2)

To analyze under which conditions these hypotheses are reasonable, consider a plane elastic medium whose dimensions and form associated with the x-y plane (denoted as plane of analysis) are arbitrary and such that the third dimension (denoted as the thickness of the piece) is associated with the z-axis (see Figure 7.1). Assume the following circumstances hold for this elastic medium:

a) The thickness \( e \) is much smaller than the typical dimension associated with the plane of analysis x-y,

\[ e \ll L. \]  \hfill (7.3)

b) The actions (body forces \( b(x, t) \), prescribed displacements \( u^*(x, t) \) and traction vector \( t^*(x, t) \)) are contained within the plane of analysis x-y (its z-component is null) and, in addition, do not depend on the third dimension,

\[
\begin{bmatrix}
\frac{b_x(x, y, t)}{0}
\end{bmatrix},
\begin{bmatrix}
0
\end{bmatrix},
\begin{bmatrix}
\frac{u^*_x(x, y, t)}{u^*_y(x, y, t)}
\end{bmatrix},
\begin{bmatrix}
\frac{t^*_x(x, y, t)}{t^*_y(x, y, t)}
\end{bmatrix},
\begin{bmatrix}
0
\end{bmatrix}
\]  \hfill (7.4)

\[ \Gamma^+ \cup \Gamma^- \cap \Gamma^\sigma : \ t^* \equiv \begin{bmatrix}
\frac{t^*_x(x, y, t)}{t^*_y(x, y, t)}
\end{bmatrix} \quad \text{null.} \]  \hfill (7.5)

c) The traction vector \( t^*(x, t) \) is only non-zero on the boundary of the piece’s thickness (boundary \( \Gamma^\sigma \)), whilst on the lateral surfaces \( \Gamma^+ \sigma \) and \( \Gamma^- \sigma \) it is null (see Figure 7.1).
Remark 7.1. The piece with the actions defined by (7.4) and (7.5) is compatible with the plane stress state given by (7.1) and (7.2), and schematized in Figure 7.2. In effect, applying the boundary conditions $\Gamma_\sigma$ on the piece yields:

- **Lateral surfaces** $\Gamma_\sigma^+$ and $\Gamma_\sigma^-$:
  \[
  n \equiv \begin{bmatrix} 0 \\ 0 \\ \pm 1 \end{bmatrix}, \quad \sigma \cdot n \equiv \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \pm 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},
  \]

- **Edge** $\Gamma_e$:
  \[
  n \equiv \begin{bmatrix} n_x \\ n_y \\ 0 \end{bmatrix}, \quad \sigma(x,y,t) \cdot n \equiv \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ 0 \end{bmatrix} = \begin{bmatrix} t_x(x,y,t) \\ t_y(x,y,t) \\ 0 \end{bmatrix},
  \]

which is compatible with the assumptions (7.4) and (7.5).

---

1 The fact that all the non-null stresses are contained in the $x$-$y$ plane is what gives rise to the name *plane stress*.
7.2.1 Strain Field. Constitutive Equation

Consider now the linear elastic constitutive equation (6.24),

\[ \varepsilon = -\frac{\nu}{E} \text{Tr}(\sigma) \mathbf{1} + \frac{1+\nu}{E} \sigma = \frac{\nu}{E} \text{Tr}(\sigma) \mathbf{1} + \frac{1}{2G} \sigma, \]

which, applied on the stress state in (7.1) and in engineering notation, provides the strains (6.25)\(^2\)

\[ \varepsilon_x = \frac{1}{E} (\sigma_x - \nu (\sigma_y + \sigma_z)) = \frac{1}{E} (\sigma_x - \nu \sigma_y) \quad \gamma_{xy} = \frac{1}{G} \tau_{xy}, \]
\[ \varepsilon_y = \frac{1}{E} (\sigma_y - \nu (\sigma_x + \sigma_z)) = \frac{1}{E} (\sigma_y - \nu \sigma_x) \quad \gamma_{xz} = \frac{1}{G} \tau_{xz} = 0, \]
\[ \varepsilon_z = \frac{1}{E} (\sigma_z - \nu (\sigma_x + \sigma_y)) = -\frac{\nu}{E} (\sigma_x + \sigma_y) \quad \gamma_{yz} = \frac{1}{G} \tau_{yz} = 0, \]

where the conditions \( \sigma_z = \tau_{xz} = \tau_{yz} = 0 \) have been taken into account. From (7.2) and (7.7) it is concluded that the strains do not depend on the \( z \)-coordinate either \( (\varepsilon = \varepsilon(x, y, t)) \). In addition, the strain \( \varepsilon_z \) in (7.7) can be solved as

\[ \varepsilon_z = -\frac{\nu}{1-\nu} (\varepsilon_x + \varepsilon_y). \]

\(^2\) The engineering angular strains are defined as \( \gamma_{xy} = 2 \varepsilon_{xy}, \gamma_{xz} = 2 \varepsilon_{xz} \) and \( \gamma_{yz} = 2 \varepsilon_{yz} \).
In short, the strain tensor for the plane stress case results in

$$\varepsilon(x, y, t) \equiv \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & 0 \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}$$

with \( \varepsilon_z = -\frac{\nu}{1-\nu}(\varepsilon_x + \varepsilon_y) \) (7.9)

and replacing (7.8) in (7.7) leads, after certain algebraic operations, to

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y), \quad \sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x),$$

and \( \tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} \),

which can be rewritten as

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \Rightarrow \{\sigma\} = \mathbf{C}^{\text{plane stress}} \cdot \{\varepsilon\}$$

(7.11)

7.2.2 Displacement Field

The components of the geometric equation of the problem (6.3),

$$\varepsilon(x, t) = \nabla^2 \mathbf{u}(x, t) = \frac{1}{2}(\mathbf{u} \otimes \nabla + \nabla \otimes \mathbf{u})$$

(7.12)

can be decomposed into two groups:

1) Those that do not affect the displacement \( u_z \) (and are hypothetically integrable in \( \mathbb{R}^2 \) for the \( x-y \) domain),

$$\varepsilon_x(x, y, t) = \frac{\partial u_x}{\partial x},$$

$$\varepsilon_y(x, y, t) = \frac{\partial u_y}{\partial y},$$

$$\gamma_{xy}(x, y, t) = 2\varepsilon_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

integration in \( \mathbb{R}^2 \)

$$\begin{cases} u_x = u_x(x, y, t) \\ u_y = u_y(x, y, t) \end{cases} \Rightarrow$$

(7.13)
2) Those in which the displacement \(u_z\) intervenes,

\[
\varepsilon_z(x,y,t) = \frac{\partial u_z}{\partial z} = -\frac{\nu}{1-\nu} (\varepsilon_x + \varepsilon_y) ,
\]

\[
\gamma_{xz}(x,y,t) = 2\varepsilon_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = 0 ,
\]

\[
\gamma_{yz}(x,y,t) = 2\varepsilon_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} = 0 .
\]

Observation of (7.1) to (7.14) suggests considering an ideal elastic plane stress problem reduced to the two dimensions of the plane of analysis and characterized by the unknowns

\[
\mathbf{u}(x,y,t) \equiv \begin{bmatrix} u_x \\ u_y \end{bmatrix} , \quad \{\varepsilon(x,y,t)\} \equiv \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \quad \text{and} \quad \{\sigma(x,y,t)\} \equiv \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} ,
\]

(7.15)
in which the additional unknowns with respect to the general problem are either null, or can be calculated in terms of those in (7.15), or do not intervene in the reduced problem,

\[
\sigma_z = \tau_{xz} = \tau_{yz} = \gamma_{xz} = \gamma_{yz} = 0 , \quad \varepsilon_z = -\frac{\nu}{1-\nu} (\varepsilon_x + \varepsilon_y) ,
\]

and \(u_z(x,y,z,t)\) does not intervene in the problem.

Remark 7.2. The plane stress problem is an ideal elastic problem since it cannot be exactly reproduced as a particular case of a three-dimensional elastic problem. In effect, there is no guarantee that the solution of the reduced plane stress \(u_x(x,y,t)\) and \(u_y(x,y,t)\) will allow obtaining a solution \(u_z(x,y,z,t)\) for the rest of components of the geometric equation (7.14).

7.3 Plane Strain

The strain state is characterized by the simplifying hypotheses

\[
\mathbf{u} \equiv \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} u_x(x,y,t) \\ u_y(x,y,t) \\ 0 \end{bmatrix} .
\]

(7.17)
Again, it is illustrative to analyze in which situations these hypotheses are plausible. Consider, for example, an elastic medium whose geometry and actions can be generated from a bidimensional section (associated with the \(x\)-\(y\) plane and with the actions \(b(x,t)\), \(u^*(x,t)\) and \(t^*(x,t)\) contained in this plane) that is translated along a straight generatrix perpendicular to said section and, thus, associated with the \(z\)-axis (see Figure 7.3).

The actions of the problem can then be characterized by

\[
\begin{align*}
\mathbf{b} &
\equiv \begin{bmatrix}
b_x(x,y,t) \\
b_y(x,y,t) \\
0
\end{bmatrix},
\Gamma_u : \mathbf{u}^* &
\equiv \begin{bmatrix}
u^*_x(x,y,t) \\
u^*_y(x,y,t) \\
0
\end{bmatrix} \text{ and } 
\Gamma_{\sigma} : \mathbf{t}^* &
\equiv \begin{bmatrix}
t^*_x(x,y,t) \\
t^*_y(x,y,t) \\
0
\end{bmatrix}.
\end{align*}
\tag{7.18}
\]

In the central section (which is a plane of symmetry with respect to the \(z\)-axis) the conditions

\[
u_z = 0 , \quad \frac{\partial u_x}{\partial z} = 0 \quad \text{and} \quad \frac{\partial u_y}{\partial z} = 0
\tag{7.19}
\]

are satisfied and, thus, the displacement field in this central section is of the form

\[
\mathbf{u}(x,y,t) &
\equiv \begin{bmatrix}
u_x(x,y,t) \\
u_y(x,y,t) \\
0
\end{bmatrix}.
\tag{7.20}
\]
7.3.1 Strain and Stress Fields

The strain field corresponding with the displacement field characteristic of a plane strain state (7.20) is

\[ \varepsilon_x(x, y, t) = \frac{\partial u_x}{\partial x}, \quad \varepsilon_y(x, y, t) = \frac{\partial u_y}{\partial y}, \quad \varepsilon_z(x, y, t) = \frac{\partial u_z}{\partial z} = 0, \]

\[ \gamma_{xy}(x, y, t) = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \quad \gamma_{xz}(x, y, t) = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = 0, \quad \gamma_{yz}(x, y, t) = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} = 0. \] (7.21)

Therefore, the structure of the strain tensor is

\[ \varepsilon(x, y, t) \equiv \begin{bmatrix} \varepsilon_x & \frac{1}{2} \gamma_{xy} & 0 \\ \frac{1}{2} \gamma_{xy} & \varepsilon_y & 0 \\ 0 & 0 & 0 \end{bmatrix}. \] (7.22)

Consider now the lineal elastic constitutive equation (6.20)

\[ \sigma = \lambda \text{Tr}(\varepsilon) \mathbf{1} + 2\mu \varepsilon = \lambda \text{Tr}(\varepsilon) \mathbf{1} + 2G\varepsilon, \] (7.23)

which, applied to the strain field (7.21), produces the stresses

\[ \sigma_x = \lambda (\varepsilon_x + \varepsilon_y) + 2\mu \varepsilon_x = (\lambda + 2G) \varepsilon_x + \lambda \varepsilon_y, \quad \tau_{xy} = G \gamma_{xy}, \]

\[ \sigma_y = \lambda (\varepsilon_x + \varepsilon_y) + 2\mu \varepsilon_y = (\lambda + 2G) \varepsilon_y + \lambda \varepsilon_x, \quad \tau_{yx} = G \gamma_{yx} = 0, \]

\[ \sigma_z = \lambda (\varepsilon_x + \varepsilon_y), \quad \tau_{xz} = G \gamma_{xz} = 0, \quad \tau_{yz} = G \gamma_{yz} = 0. \] (7.24)

Considering (7.21) and (7.24), one concludes that stresses do not depend on the \( z \)-coordinate either \( (\sigma = \sigma(x, y, t)) \). On the other hand, the stress \( \sigma_z \) in (7.24) can be solved as

\[ \sigma_z = \frac{\lambda}{2(\lambda + \mu)} (\sigma_x + \sigma_y) = \nu (\sigma_x + \sigma_y) \] (7.25)

3 By analogy with the plane stress case, the fact that all non-null strains are contained in the \( x-y \) plane gives rise to the name plane strain.
and the stress tensor for the plane strain case results in

\[
\sigma(x, y, t) \equiv \begin{bmatrix}
\sigma_x & \tau_{xy} & 0 \\
\tau_{xy} & \sigma_y & 0 \\
0 & 0 & \sigma_z
\end{bmatrix}
\]

with \( \sigma_z = -\nu(\sigma_x + \sigma_y) \), (7.26)

where the non-null components of the stress tensor (7.26) are

\[
\sigma_x = (\lambda + 2G)\varepsilon_x + \lambda \varepsilon_y = \frac{E(1 - \nu)}{(1 + \nu)(1 - 2\nu)} \left( \varepsilon_x + \frac{\nu}{1 - \nu} \varepsilon_y \right),
\]

\[
\sigma_y = (\lambda + 2G)\varepsilon_y + \lambda \varepsilon_x = \frac{E(1 - \nu)}{(1 + \nu)(1 - 2\nu)} \left( \varepsilon_y + \frac{\nu}{1 - \nu} \varepsilon_x \right),
\]

and \( \tau_{xy} = G\gamma_{xy} = \frac{E}{2(1 + \nu)}\gamma_{xy} \).

Equation (7.27) can be rewritten in matrix form as

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} = C_{\text{plane strain}} \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} \Rightarrow \left\{ \sigma \right\} = C_{\text{plane strain}} \cdot \left\{ \varepsilon \right\}.
\]

Equation (7.28)

Similarly to the plane stress problem, (7.20), (7.21) and (7.26) suggest considering an elastic plane strain problem reduced to the two dimensions of the plane of analysis \( x - y \) and characterized by the unknowns

\[
\mathbf{u}(x, y, t) \equiv \begin{bmatrix} u_x \\
u_y \end{bmatrix}, \left\{ \mathbf{\varepsilon}(x, y, t) \right\} \equiv \begin{bmatrix} \varepsilon_x \\
\varepsilon_y \\
\gamma_{xy} \end{bmatrix} \text{ and } \left\{ \mathbf{\sigma}(x, y, t) \right\} \equiv \begin{bmatrix} \sigma_x \\
\sigma_y \\
\tau_{xy} \end{bmatrix}, \]

(7.29)
in which the additional unknowns with respect to the general problem are either null or can be calculated in terms of those in (7.29),

\[ u_z = 0, \quad \varepsilon_z = \gamma_{xz} = \gamma_{yz} = \tau_{xz} = \tau_{yz} = 0 \quad \text{and} \quad \sigma_z = \nu (\sigma_x + \sigma_y). \]  

(7.30)

### 7.4 The Plane Linear Elastic Problem

In view of the equations in Sections 7.2 and 7.3, the linear elastic problem for the plane stress and plane strain problems is characterized as follows (see Figure 7.4).

**Equations**

a) Cauchy’s equation

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho b_x = \rho \frac{\partial^2 u_x}{\partial t^2} \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \rho b_y = \rho \frac{\partial^2 u_y}{\partial t^2}
\]  

(7.31)

---

4 The equation corresponding to the z-component either does not intervene (plane stress), or is identically null (plane strain).
b) Constitutive equation

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix}
\equiv
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
\quad \text{and} \quad
\{\sigma\} = \mathbf{C} \cdot \{\varepsilon\},
\]

(7.32)

where the constitutive matrix \(\mathbf{C}\) can be written in a general form, from (7.11) and (7.28), as

\[
\mathbf{C} \equiv \frac{E}{1 - \nu^2}
\begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1 - \nu}{2}
\end{bmatrix}
\]

Plane stress

\[
\begin{align*}
\dot{E} &= E \\
\dot{\nu} &= \nu
\end{align*}
\]

(7.33)

Plane strain

\[
\begin{align*}
\dot{E} &= E \\
\dot{\nu} &= \nu
\end{align*}
\]

(7.33)

c) Geometric equation

\[
\varepsilon_x = \frac{\partial u_x}{\partial x}, \quad \varepsilon_y = \frac{\partial u_y}{\partial y}, \quad \gamma_{xy} = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}
\]

(7.34)

d) Boundary conditions in space

\[
\Gamma_u: \quad u_x \equiv \begin{bmatrix} u_x(x,y,t) \\ u_y(x,y,t) \end{bmatrix}, \quad \Gamma_\sigma: \quad t^* \equiv \begin{bmatrix} t_x^*(x,y,t) \\ t_y^*(x,y,t) \end{bmatrix}
\]

(7.35)

\[
t^* = \sigma \cdot n, \quad \sigma \equiv \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}, \quad n \equiv \begin{bmatrix} n_x \\ n_y \end{bmatrix}
\]

e) Initial conditions

\[
\begin{align*}
\begin{bmatrix} u(x,y,t) \\ \dot{u}(x,y,t) \end{bmatrix} \bigg|_{t=0} &= \mathbf{0}, \quad \begin{bmatrix} u(x,y,t) \\ \dot{u}(x,y,t) \end{bmatrix} \bigg|_{t=0} = \mathbf{v}_0(x,y)
\end{align*}
\]

(7.36)
Unknowns

\[ \mathbf{u}(x, y, t) \equiv \begin{bmatrix} u_x \\ u_y \end{bmatrix}, \ \boldsymbol{\varepsilon}(x, y, t) \equiv \begin{bmatrix} \varepsilon_x & \frac{1}{2} \gamma_{xy} \\ \frac{1}{2} \gamma_{xy} & \varepsilon_y \end{bmatrix}, \ \boldsymbol{\sigma}(x, y, t) \equiv \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \] (7.37)

Equations (7.31) to (7.37) define a system of 8 PDEs with 8 unknowns that must be solved in the reduced space-time domain \( \mathbb{R}^2 \times \mathbb{R}^+ \). Once the problem is solved, the following can be explicitly calculated:

Plane stress \( \rightarrow \varepsilon_z = \frac{v}{1-v} (\varepsilon_x + \varepsilon_y) \) (7.38)
Plane strain \( \rightarrow \sigma_z = v (\sigma_x + \sigma_y) \)

7.5 Problems Typically Assimilated to Plane Elasticity

7.5.1 Plane Stress

The stress and strain states produced in solids that have a dimension considerably inferior to the other two (which constitute the plane of analysis x-y) and whose actions are contained in said plane are typically assimilated to a plane stress state. The slab loaded on its mean plane and the deep beam of Figure 7.5 are classic examples of structures that can be analyzed as being in a plane stress state. As a particular case, the problems of simple and complex bending in beams considered in strength of materials can also be assimilated to plane stress problems.

Figure 7.5: Slab loaded on its mean plane (left) and deep beam (right).
7.5.2 Plane Strain

The solids whose geometry can be obtained by translation of a plane section with actions contained in its plane (plane of analysis \(x-y\)) along a generatrix line perpendicular to said section are typically assimilated to plane strain states. In addition, the plane strain hypothesis \(\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0\) must be justifiable. In general, this situation occurs in two circumstances:

1) The dimension of the piece in the \(z\)-direction is very large (for the purposes of analysis, it is assumed to be infinite). In this case, any central transversal section (not close to the extremes) can be considered a symmetry plane and, thus, satisfies the conditions

\[
\begin{align*}
    u_z &= 0, \\
    \frac{\partial u_x}{\partial z} &= 0 \quad \text{and} \quad \frac{\partial u_y}{\partial z} = 0,
\end{align*}
\]

which result in the initial condition of the plane strain state (7.17),

\[
\mathbf{u} \equiv \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} u_x (x, y, t) \\ u_y (x, y, t) \\ 0 \end{bmatrix}.
\]

Examples of this case are a pipe under internal (and/or external) pressure (see Figure 7.6), a tunnel (see Figure 7.7) and a strip foundation (see Figure 7.8).

2) The length of the piece in the longitudinal direction is reduced, but the displacements in the \(z\)-direction are impeded by the boundary conditions at the end sections (see Figure 7.9).

In this case, the plane strain hypothesis (7.17) can be assumed for all the transversal sections of the piece.
Figure 7.7: Tunnel.

Figure 7.8: Strip foundation.

Figure 7.9: Solid with impeded $z$-displacements.
7.6 Representative Curves of Plane Elasticity

There is an important tradition in engineering of graphically representing the distribution of plane elasticity. To this aim, certain families of curves are used, whose plotting on the plane of analysis provides useful information of said stress state.

7.6.1 Isostatics or stress trajectories

**Definition 7.1.** The isostatics or stress trajectories are the envelopes of the vector field determined by the principal stresses.

Considering the definition of the envelope of a vector field, isostatics are, at each point, tangent to the two principal directions and, thus, there exist two families of isostatics:

- Isostatics $\sigma_1$, tangent to the direction of the largest principal stress.
- Isostatics $\sigma_2$, tangent to the direction of the smallest principal stress.

In addition, since the principal stress directions are orthogonal to each other, both families of curves are also be orthogonal. The isostatic lines provide information on the mode in which the flux of principal stresses occurs on the plane of analysis.

As an example, Figure 7.10 shows the distribution of isostatics on a supported beam with uniformly distributed loading.

**Definition 7.2.** A singular point is a point characterized by the stress state

$$\sigma_x = \sigma_y \quad \text{and} \quad \tau_{xy} = 0$$

and its Mohr's circle is a point on the axis $\sigma$ (see Figure 7.11).

A neutral point is a singular point characterized by the stress state

$$\sigma_x = \sigma_y = \tau_{xy} = 0$$

and its Mohr's circle is the origin of the $\sigma - \tau$ space (see Figure 7.11).
Remark 7.3. All directions in a singular point are principal stress directions (the pole is the Mohr’s circle itself, see Figure 7.11). Consequently, the isostatics tend to lose their regularity in singular points and can brusquely change their direction.

7.6.1.1 Differential Equation of the Isostatics

Consider the general equation of an isostatic line $y = f(x)$ and the value of the angle formed by the principal stress direction $\sigma_1$ with respect to the horizontal direction (see Figure 7.12),

$$\tan(2\alpha) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2\tan \alpha}{1 - \tan^2 \alpha} \quad \Rightarrow \quad \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2y'}{1 - (y')^2} \quad \Rightarrow$$

$$(y')^2 + \frac{\sigma_x - \sigma_y}{\tau_{xy}} y' - 1 = 0 \quad (7.41)$$
and solving the second-order equation (7.41) for \( y' \), the differential equation of the isostatics is obtained.

\[
\text{Differential equation of the isostatics} \quad \Rightarrow \quad y' = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2\tau_{xy}}\right)^2 + 1} \phi(x,y) \tag{7.42}
\]

If the function \( \phi(x,y) \) in (7.42) is known, this equation can be integrated to obtain the algebraic equation of the family of isostatics,

\[
y = f(x) + C. \tag{7.43}
\]

The double sign in (7.42) leads to two differential equations corresponding to the two families of isostatics.

**Example 7.1** – A rectangular plate is subjected to the following stress states.

\[
\sigma_x = -x^3; \quad \sigma_y = 2x^3 - 3xy^2; \quad \tau_{xy} = 3x^2y; \quad \tau_{xz} = \tau_{yz} = \sigma_z = 0
\]

Obtain and plot the singular points and distribution of isostatics.

**Solution**

The singular points are defined by \( \sigma_x = \sigma_y \) and \( \tau_{xy} = 0 \). Then,

\[
\tau_{xy} = 3x^2y = 0 \quad \Rightarrow \quad \begin{cases} x = 0 & \Rightarrow \begin{cases} \sigma_x = -x^3 = 0 \\ \sigma_y = 2x^3 - 3xy^2 = 0 \end{cases} \\ y = 0 & \Rightarrow \begin{cases} \sigma_x = -x^3 \\ \sigma_y = 2x^3 - 3xy^2 = 2x^3 \end{cases} \end{cases} \quad \Rightarrow \quad x = 0
\]
Therefore, the locus of singular points is the straight line $x = 0$. These singular points are, in addition, neutral points ($\sigma_x = \sigma_y = 0$).

The *isostatics* are obtained from (7.42),

$$y' = \frac{dy}{dx} = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2\tau_{xy}}\right)^2 + 1},$$

which, for the given data of this problem, results in

$$\begin{cases}
\frac{dy}{dx} = \frac{x}{y} \\
\frac{dy}{dx} = \frac{-y}{x}
\end{cases} \implies \text{integrating} \implies \begin{cases}
x^2 - y^2 = C_1 \\
x y = C_2
\end{cases}.$$ 

Therefore, the isostatics are two families of equilateral hyperbolas orthogonal to each other.

On the line of singular points $x = 0$ (which divides the plate in two regions) the isostatics will brusquely change their slope. To identify the family of isostatics $\sigma_1$, consider a point in each region:

- Point $(1,0)$: $\sigma_x = \sigma_2 = -1; \sigma_y = \sigma_1 = +2; \tau_{xy} = 0$ (isostatic $\sigma_1$ in the $y$-direction)
- Point $(-1,0)$: $\sigma_x = \sigma_1 = +1; \sigma_y = \sigma_2 = -2; \tau_{xy} = 0$ (isostatic $\sigma_1$ in the $x$-direction)

Finally, the distribution of isostatics is as follows:

![Diagram of isostatics]

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7.6.2 Isoclines

**Definition 7.3.** Isoclines are the locus of the points in the plane of analysis along which the principal stress directions form a certain angle with the $x$-axis.

It follows from its definition that in all the points of a same isocline the principal stress directions are parallel to each other, forming a constant angle $\theta$ (which characterizes the isocline) with the $x$-axis (see Figure 7.13).

7.6.2.1 Equation of the Isoclines

The equation $y = f(x)$ of the isocline with an angle $\theta$ is obtained by establishing that the principal stress direction $\sigma_1$ forms an angle $\alpha = \theta$ with the horizontal direction, that is,

$$
\tan(2\theta) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}
$$

(7.44)

This algebraic equation allows isolating, for each value of $\theta$,

$$
y = f(x, \theta),
$$

(7.45)

which constitutes the equation of the family of isoclines parametrized in terms of the angle $\theta$.

Figure 7.13: Isocline.
Remark 7.4. Determining the family of isoclines allows knowing, at each point in the medium, the direction of the principal stresses and, thus, the obtainment of the isostatics can be sought. Given that isoclines can be determined by means of experimental methods (methods based on photoelasticity) they provide, indirectly, a method for the experimental determination of the isostatics.

7.6.3 Isobars

Definition 7.4. Isobars are the locus of points in the plane of analysis with the same value of principal stress $\sigma_1$ (or $\sigma_2$).

Two families of isobars will cross at each point of the plane of analysis: one corresponding to $\sigma_1$ and another to $\sigma_2$. Note that the isobars depend on the value of $\sigma_1$, but not on its direction (see Figure 7.14).

7.6.3.1 Equation of the Isobars

The equation that provides the value of the principal stresses (see Chapter 4) implicitly defines the algebraic equation of the two families of isobars $y = f_1(x, c_1)$ and $y = f_2(x, c_2)$.

\[
\begin{align*}
\sigma_1 &= \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \text{const.} = c_1 \\
\phi_1(x, y) &
\end{align*}
\]

\[
\begin{align*}
\sigma_2 &= \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \text{const.} = c_2 \\
\phi_2(x, y) &
\end{align*}
\]

which leads to

\[
\begin{align*}
y &= f_1(x, c_1) \\
y &= f_2(x, c_2)
\end{align*}
\]
7.6.4 Maximum Shear Stress or Slip Lines

Definition 7.5. Maximum shear stress lines or slip lines are the envelopes of the directions that, at each point, correspond with the maximum value (in modulus) of the shear (or tangent) stress.

Remark 7.5. At each point of the plane of analysis there are two planes on which the shear stresses reach the same maximum value (in module) but that have opposite directions, \( \tau_{\text{max}} \) and \( \tau_{\text{min}} \). These planes can be determined by means of the Mohr’s circle and form a 45° angle with the principal stress directions (see Figure 7.15). Therefore, their envelopes (maximum shear stress lines) are two families of curves orthogonal to each other that form a 45° angle with the isostatics.

7.6.4.1 Differential Equation of the Maximum Shear Lines

Consider \( \beta \) is the angle formed by the direction of \( \tau_{\text{max}} \) with the horizontal direction (see Figure 7.16). In accordance with Remark 7.5\(^5\),

\[
\beta = \alpha - \frac{\pi}{4} \implies \tan(2\beta) = \tan\left(2\alpha - \frac{\pi}{2}\right) = -\frac{1}{\tan(2\alpha)}, \quad (7.48)
\]

\(^5\) Here, the trigonometric expression \( \tan(\theta - \pi/2) = -\cot \theta = -1/\tan \theta \) is used.
where $\alpha$ is the angle formed by the principal stress direction $\sigma_1$ with the horizontal direction. Consequently, considering the general equation of a slip line, $y = f(x)$, the expression (7.48) and the relation $\tan(2\alpha) = 2\tau_{xy}/(\sigma_x - \sigma_y)$ yields

$$\tan(2\beta) = -\frac{1}{\tan(2\alpha)} = \frac{\sigma_x - \sigma_y}{2\tau_{xy}} \left(\frac{2\tan\beta}{1 - \tan^2\beta}\right) \Rightarrow$$

$$\tan\beta = \frac{dy}{dx} = y'$$

$$-\frac{\sigma_x - \sigma_y}{2\tau_{xy}} = 1 - (y')^2$$

$$\Rightarrow (y')^2 - \frac{4\tau_{xy}}{\sigma_x - \sigma_y} y' - 1 = 0.$$  

Solving the second-order equation in (7.49) for $y'$ provides the differential equation of the maximum shear stress lines.

**Differential equation of the max. shear stress or slip lines**

$$y' = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \pm \sqrt{\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)^2 + 1} \varphi(x,y)$$  

(7.50)

If the function $\varphi(x,y)$ in (7.50) is known, this differential equation can be integrated and the algebraic equation of the two families of orthogonal curves (corresponding to the double sign in (7.50)) is obtained.
Figure 7.16: Maximum shear stress or slip lines.
PROBLEMS

Problem 7.1 – Justify whether the following statements are true or false.

a) If a plane stress state has a singular point, all the isoclines cross this point.
b) If a plane stress state is uniform, all the slip lines are parallel to each other.

Solution

a) A singular point is defined as:

\[
\begin{align*}
\sigma_1 &= \sigma_2 \\
\tau &= 0
\end{align*}
\]

Therefore, all directions are principal stress directions and, given an angle \( \theta \) which can take any value, the principal stress direction will form an angle \( \theta \) with the x-axis. Then, an isocline of angle \( \theta \) will cross said point and, since this holds true for any value of \( \theta \), all the isoclines will cross this point. Therefore, the statement is true.

b) A uniform stress state implies that the Mohr’s circle is equal in all points of the medium, therefore, the planes of maximum shear stress will be the same in all points. Then, the maximum shear stress lines (or slip lines) will be parallel to each other. In conclusion, the statement is true.
Problem 7.2 – A rectangular plate is subjected to the following plane stress states.

1) \( \sigma_x = 0 ; \sigma_y = b > 0 ; \tau_{xy} = 0 \)
2) \( \sigma_x = 0 ; \sigma_y = 0 ; \tau_{xy} = my, \ m > 0 \)

Plot for each state the isostatics and the slip lines, and indicate the singular points.

Solution

1) The Mohr’s circle for the stress state \( \sigma_x = 0 ; \sigma_y = b > 0 ; \tau_{xy} = 0 \) is:

Then, the isostatics are:

Isostatics \( \sigma_f \)
Isostatics \( \sigma_{II} \)
And the *slip lines* are:

There do not exist singular points for this stress state.

2) The Mohr’s circle for the stress state $\sigma_x = 0; \sigma_y = 0; \tau_{xy} = my, \ m > 0$ is:
Then, the *isostatics* and singular points are:

![Isostatics diagram](image1)

And the *slip lines* are:

![Slip lines diagram](image2)
EXERCISES

7.1 – A rectangular plate is subjected to the following plane strain state:

\[
\begin{align*}
\sigma_x &= \sigma_y \\
\tau_{xy} &= ax \\
\sigma_y &= b
\end{align*}
\]

\((a > 0, b > 0)\)

Plot the isostatics and the slip lines, and indicate the singular points.

7.2 – Plot the isostatics in the transversal section of the cylindrical shell shown below. Assume a field of the form:

\[
\begin{align*}
ur &= Ar + \frac{B}{r} ; \quad A > 0, B > 0 \\
u_{\theta} &= 0 \\
z &= 0
\end{align*}
\]